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Abstract The spectrum of modes associated with a magnetic X–point is determined, taking into account resistivity and electron inertia in Ohm's law. The assumed equilibrium configuration has zero current. In the limit of zero resistivity and viscosity, the equation describing perturbations to the X–point equilibrium has two classes of continuum eigensolutions. We find that in the case of finite resistivity and zero viscosity, there are both continuous and discrete spectra present depending upon the relative values of the resistive length scale and the collisionless skin depth. The continuum is associated with singular eigenmodes. The magnetic field then has a logarithmic singularity and finite energy, while the perturbed velocity is not square-integrable and the kinetic energy is thus infinite. The current density has both even and odd parity current sheets about the singularity. In the absence of viscosity, for a given ratio of collisionless skin depth to system size, there is an upper limit to the real eigenmode frequency; there is no such upper limit in the purely ideal or resistive magnetohydrodynamics (MHD) cases. The damping rate of the continuum eigenmodes scales with the square of the ratio of the resistive length to the collisionless skin depth, and is independent of the frequency. This damping is much weaker than that predicted by resistive MHD in the large Lundquist number limit. The analysis shows that electron inertia does not automatically resolve singularities of ideal MHD. Indeed electron MHD, unlike purely resistive MHD which only has a discrete spectrum of strongly damped eigenmodes, permits the existence of weakly damped eigenmodes (discrete or continuum), with frequencies typically in the Alfvén range, which could be readily excited by other instabilities. Such modes could redistribute or accelerate energetic ions and possibly affect turbulent plasma transport in the vicinity of X-points. The results apply both to azimuthally–symmetric modes and to more general helical modes with nonzero azimuthal and longitudinal wave numbers. In the presence of a small electron viscosity, the singularities are resolved and the eigenmode spectrum again becomes purely discrete, involving viscous critical layers.

1 Introduction

Magnetic X–points exist close to the plasma boundary of divertor tokamaks. In tokamak plasmas generally, X–points are produced by reconnection of magnetic field lines resulting from magnetohydrodynamic (MHD) instabilities. The widths of magnetic islands formed by reconnection can be amplified as a result of the removal of bootstrap current from the vicinity of the island O–points. These "neoclassical tearing modes" can limit the achievable plasma pressure and hence fusion yield in large tokamaks. In addition, "classical tearing modes", involving reconnecting nonlinearly saturated magnetic island structures [1, 2] driven by current gradients, can also occur. The latter are thought to be responsible for sawteeth (with dominant poloidal and toroidal mode numbers $m = 1, n = 1$), major disruptions ($m = 2, n = 1$) and possibly also edge localised modes (ELMs). The basic properties of magnetic X–points are of intrinsic plasma physics interest. The fact that they are associated with performance–limiting instabilities in magnetic fusion experiments creates an additional motivation to study them.

In a paper focused on the problem of energy release in solar flares, Craig and McClymont [3] studied small amplitude oscillations associated with a current–free two–dimensional X–point in the limit of incompressible resistive MHD. Specifically, they considered an equilibrium magnetic field of the form

$$\mathbf{B}_E = \frac{B_0}{R_0}(y\hat{\mathbf{x}} + x\hat{\mathbf{y}}) \tag{1}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ denote unit vectors in the x and y directions and B_0 is the field at a circular boundary, $R = (x^2 + y^2)^{1/2} = R_0$. This field can be written as $\nabla \times (\psi_E \hat{\mathbf{z}})$, where $\hat{\mathbf{z}}$ denotes the unit vector in the z direction and

$$\psi_E = \frac{1}{2}\frac{B_0}{R_0}(y^2 - x^2) \tag{2}$$

It follows immediately from Ampère's law that such a configuration has zero current. Craig and McClymont considered incompressible, inviscid perturbations to this equilibrium in the (x, y) plane. They showed that only azimuthally symmetric perturbations are associated with true magnetic reconnection. Because dissipation is taken into account, the normal modes of the system have complex eigenvalues; since the equilibrium configuration is potential, all the modes are damped. Hamilton and co–workers [4] have demonstrated recently that the perturbation analysis of Craig and McClymont remains valid if there is a constant and uniform magnetic field component in the z–direction, provided that any plasma flows are restricted to the (x, y) plane.

Craig and McClymont used the resistive MHD form of Ohm's law:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}, \tag{3}$$

where $\mathbf{E}$ is the electric field, $\mathbf{v}$ is the flow velocity, $\mathbf{j}$ is current density and η is resistivity, assumed constant. For certain applications, such as sawtooth relaxation events in high temperature tokamak discharges [5, 6], it is appropriate to use a more general form of Ohm's law in which electron inertia is taken into account:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j} + \frac{m_e}{ne^2} \frac{\partial \mathbf{j}}{\partial t}, \tag{4}$$

where m_e, e, n denote respectively the electron mass, charge and density (assumed constant). In this paper we extend the analysis of Craig and McClymont to determine the mode spectrum of a system with equilibrium configuration in the (x, y) plane given by Eq. (1), and with Ohm's law taking the form of Eq. (4) rather than Eq. (3). More general forms of Ohm's law also includes pressure gradient terms, the Hall current term, and a term associated with viscosity of the electron fluid; we neglect pressure gradient terms in both Ohm's law and the fluid momentum equation, but later in the paper we discuss the possible effects of finite electron viscosity on the X–point eigenvalue spectrum.

There are several motivations for this study. Normal modes associated with X–points constitute a possible channel for the dissipation of any free energy in the system, and could thus play a role in the time evolution of the X–point configuration. If detected by magnetic coils, the modes could in principle provide diagnostic information on plasma parameters, as in the case of other high frequency Alfvénic instabilities [7]. Secondly, as suggested in the case of solar flares [4], the modes could accelerate or redistribute charged particles in the X–point vicinity. Finally, if driven by other instabilities nonlinearly, the modes could play an important role in transport processes near X-points.

The paper is organized as follows. After formulating the general eigenvalue equation in Section 2, including the effects of electron viscosity, we study analytically the properties of the inviscid equation, first in the limit of zero resistivity (Section 3) then for the case of finite resistivity (Section 4). Computations of eigenvalues and eigenfunctions, carried out using two complementary techniques, are presented in Section 5, and in Section 6 we demonstrate that singularities in the inviscid problem are eliminated by electron viscosity. In the final section we present a brief discussion of the significance of the results and some conclusions.

2 Formulation of Eigenvalue Problem

With the electron inertia term included in Ohm's law, it is straightforward to show that $\mathbf{B}$ satisfies the evolution equation

$$\frac{\partial}{\partial t} \left(\mathbf{B} - \frac{c^2}{\omega_{pe}^2} \nabla^2 \mathbf{B} \right) = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{\eta}{\mu_0} \nabla^2 \mathbf{B}, \tag{5}$$

where c is the speed of light, ω_{pe} is the electron plasma frequency and μ_0 is the permeability of free space. Writing

$$\mathbf{B} = \nabla \times (\psi \hat{\mathbf{z}}) + B_z \hat{\mathbf{z}}, \tag{6}$$

where B_z is a constant, Eq. (5) becomes

$$\frac{\partial}{\partial t}\left(\psi - \frac{c^2}{\omega_{pe}^2}\nabla^2\psi\right) + (\mathbf{v}\cdot\nabla)\,\psi = \frac{\eta}{\mu_0}\nabla^2\psi. \tag{7}$$

Neglecting plasma pressure and viscosity, the appropriate form of the momentum equation is

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = \frac{1}{\rho}\mathbf{j}\times\mathbf{B}, \tag{8}$$

where ρ is mass density. It is easily verified that $\mathbf{v} = 0$, $\psi = \psi_E$ define a steady–state solution of Eqs. (7) and (8). Putting $\psi = \psi_E + \tilde{\psi}$, where $|\nabla\tilde{\psi}| \ll |\nabla\psi_E|$, and using the fact that the equilibrium has zero flow (so that $\mathbf{v}$ can be treated as a perturbation), it is straightforward to show that Eqs. (7) and (8), when linearized, can be combined into a single equation for $\tilde{\psi}$:

$$\frac{\partial^2}{\partial t^2}\left(\tilde{\psi} - \frac{c^2}{\omega_{pe}^2}\nabla^2\tilde{\psi}\right) - \frac{B_0^2}{\mu_0\rho}\frac{R^2}{R_0^2}\nabla^2\tilde{\psi} - \frac{\eta}{\mu_0}\nabla^2\frac{\partial\tilde{\psi}}{\partial t} = 0. \tag{9}$$

To obtain this result we have used the fact that $|\nabla\psi_E|^2 = B_0^2R^2/R_0^2$. In view of the fact that the equilibrium magnetic configuration [Eq. (2)] depends on the azimuthal angle $\varphi \equiv \tan^{-1}(y/x)$, it is remarkable that φ does not appear in the equation for the perturbed fields. This feature allows separability of the eigenmodes.

It is appropriate at this point to introduce dimensionless variables. Following Ref. [3], we write $r = R/R_0$ and normalize time to R_0/c_{A0} where $c_{A0} = B_0/\sqrt{\mu_0\rho}$ is the Alfvén speed at $R = R_0$ defined in terms of the magnetic field in the (x, y) plane. In applying the model to, for example, magnetic islands in tokamaks, the boundary R_0 should be chosen to be smaller than the island width. We also introduce $S = \mu_0 R_0 c_{A0}/\eta$, the Lundquist number at $R = R_0$, and the dimensionless electron skin depth $\delta_e = c/(\omega_{pe}R_0)$. The typical resistive length scale is $L_\eta = R_0 S^{-1/2}$. It follows that $c/(\omega_{pe}L_\eta) = \delta_e S^{1/2}$. This nondimensional ratio will play an important role in the later analysis. Written in terms of these dimensionless variables and parameters, Eq. (9) becomes

$$\frac{\partial^2}{\partial t^2}\left(\tilde{\psi} - \delta_e^2\nabla^2\tilde{\psi}\right) - r^2\nabla^2\tilde{\psi} - \frac{1}{S}\nabla^2\frac{\partial\tilde{\psi}}{\partial t} = 0. \tag{10}$$

The Laplacian operator in Eq. (10) can be written as

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}, \tag{11}$$

We now seek separable solutions for $\tilde{\psi}$ of the form

$$\tilde{\psi} = e^{i(m\varphi+kz-\omega t)} f(r). \tag{12}$$

The Laplacian then becomes a differential operator in r:

$$\nabla^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) - (\frac{m^2}{r^2} + k^2) \tag{13}$$

Equation (10) then reduces to

$$\omega^2 f + \left(r^2 - \omega^2\delta_e^2 - i\frac{\omega}{S}\right)\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - (\frac{m^2}{r^2} + k^2)f\right] = 0. \tag{14}$$

Instead of Eq. (4), one may use a more general Ohm's law that includes a simple formulation of classical electron perpendicular viscosity but neglects parallel viscosity [8]:

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta\mathbf{j} + \frac{m_e}{ne^2}\left\{\frac{\partial\mathbf{j}}{\partial t} - \frac{\rho_e^2}{\tau_e}\nabla^2\mathbf{j}\right\}, \tag{15}$$

where ρ_e is thermal electron Larmor radius and τ_e is electron collision time. It is straightforward to show that Eq. (9) becomes

$$\frac{\partial^2}{\partial t^2}\left(\tilde{\psi} - \frac{c^2}{\omega_{pe}^2}\nabla^2\tilde{\psi}\right) - \frac{B_0^2}{\mu_0\rho}\frac{R^2}{R_0^2}\nabla^2\tilde{\psi} - \frac{\eta}{\mu_0}\nabla^2\frac{\partial\tilde{\psi}}{\partial t} = -\frac{c^2\rho_e^2}{\omega_{pe}^2\tau_e}\nabla^4\frac{\partial\tilde{\psi}}{\partial t}. \tag{16}$$

Introducing a dimensionless electron Larmor radius $\rho_{*e} = \rho_e/R_0$ and a dimensionless collision time $\tau_{*e} = R_0\tau_e/c_{A0}$, we deduce the following eigenvalue equation

$$\omega^2 f + \left(r^2 - \omega^2\delta_e^2 - i\frac{\omega}{S}\right)\nabla^2 f = -i\omega\delta_e\epsilon\nabla^4 f, \tag{17}$$

where $\epsilon \equiv (\delta_e\rho_{*e}^2/\tau_{*e}^2)$ is the electron perpendicular viscosity Reynolds number .

3 Spectral Analysis Part I: $S \to \infty, \epsilon \to 0$

Before we proceed to analyse the eigenvalue problem formulated in the previous Section, it is worthwhile to establish some standard terminology associated with eigenvalue problems. Considering Eq. (17), let us note that S, δ_e, ϵ, m and k are to be regarded as specified parameters. The first three are given by the physical conditions at hand, whilst the last two are wavenumbers, as indicated by Eq. (12). In general, we seek a nontrivial solution of the equation subject to the specified boundary data.

It is well-known that such solutions can exist only for certain values of the generally complex parameter ω. The set of values of this eigenparameter in the complex plane for which nontrivial solutions exist is called the *spectrum* of the eigenvalue problem. The spectrum is always a closed, proper subset of the complex plane. For each value in this set, there may be one or more (in cases of degeneracy) eigenmodes (i.e. complex functions $f(S, \delta_e, \epsilon, m, k; \omega; r)$ which satisfy the equation and the imposed boundary conditions). If *all* such eigenmodes and the spectrum can be determined, it is then known from the general theory of such problems [9] that the solution of the initial-boundary value problem for Eq. (16) can be solved by expanding the specified initial function in terms of the complete set of eigenmodes. The aim of the spectral theory of equations such as Eq. (17) is therefore a) to determine the spectrum; b) to determine the associated eigenmodes and c) to establish an "expansion theorem" which enables the solution of the initial value problem in terms of the eigenfunctions. Such an expansion is commonly called the "spectral resolution" of the problem and is exactly analogous to solving equations with constant coefficients by Fourier-Laplace transforms. The expansion problem is essentially a technical matter (given the spectrum and the eigenfunctions) which is best handled by constructing the Green's function for the system in terms of the eigenmodes, as demonstrated for example by Titchmarsh [9]. Indeed, the Green's function method is so powerful that it can be directly used to actually construct the eigenfunctions and eigenvalues, as shown in [9, 10, 12, 13].

We shall be exclusively concerned in this paper with the first two tasks outlined in the previous paragraph. It should be stressed that in general, the spectrum (for differential or integral operators) must consist of an infinity of points in the complex ω-plane. It is also the case that the spectrum, at least for equations of the type considered here, may consist of two topologically distinct types of point sets. Thus, the *discrete* spectrum is composed of points which are isolated in the sense that each will have a neighbourhood where there will be no other eigenvalues, and may form a finite (possibly empty) or a countably infinite set; the *continuous* spectrum (when it exists) will consist of *continua* which may be curves or segments of curves in the complex plane. Discrete eigenvalues occur at the poles of the Green's function the residues of which give the eigenmodes. The continua are usually associated with branch point singularities of the Green's function and its discontinuities across suitable branch cuts in the complex ω-plane[9].

The delineation of these different possibilities is of great importance for applications, as first demonstrated by Case [10]. The works of Titchmarsh and Case show that the continuum eigenmodes are always associated with a singularity of the problem, be it due to infinite extent of the solution domain (e.g. plane waves and Bloch waves in quantum mechanics, the Rydberg continuum of the hydrogen atom) or to singularities in the coefficients in the equations (e.g. inviscid Couette flow considered by Case [10], the well-known Alfvén continuum in ideal MHD, the Vlasov problem of collisonless kinetics) which often have a clear physical origin and sometimes a physical resolution in terms of higher order effects not included in the eigenvalue problem considered (for some elementary explicit examples relevant to plasma physics, see [11, 12, 13]). Our purpose in this paper is to describe in analytic and numerical terms the complete spectral theory of the problems posed in the previous section and exhibit its stuctural richness and possible physical significance for more complex problems.

We begin the spectral theory with an important special case: we assume that $S \to \infty$ in the above equations. If we neglect electron perpendicular viscosity the motion is governed solely by electron inertia, a time-reversible, reactive effect. It should be noted, however, that electron inertial terms in Ohm's Law can cause collisionless reconnection. We identify two separate cases: Case 1, in which $m = 0, k = 0$ (where m, k are the wave numbers associated with ϕ, z respectively); and Case 2 where m, k do not vanish. Case 1 is considered in detail. We shall see later that Case 2 can be analyzed in exactly the same way.

The eigenvalue equation is reduced in Case 1 by making the substitution $x = r^2$; we then find $\nabla^2 = 4\frac{d}{dx}(x\frac{d}{dx}) \equiv 4D^2$ and

$$\omega^2 f + (x - \omega^2\delta_e^2)4D^2 f \quad = \quad -i\omega\delta_e\epsilon 16 D^4 f \tag{18}$$

This case has the following boundary conditions: at $x = 0$ we must have f regular (or analytic). This requires df/dx to be finite (although the perturbed azimuthal field amplitude, which is proportional to df/dr, is zero). We also require $d^2f/dx^2 = 0$ at the origin. At the right hand end point $x = 1$ (in our nondimensional units), we have $df/dx = d^2f/dx^2 = 0$. These conditions are of the Neumann type in that the value of f can be altered by an arbitrary constant. This is as it should be in that only the spatial derivatives of f represent physical quantities such as magnetic field and current density. We now have a well-posed, two-point eigenvalue problem for the fourth order system.

We note that for $\epsilon \neq 0$ all the coefficients in Eq. (18) are analytic in the parameters and x. Note that the coefficient of the highest order spatial derivative does not vanish. Under these circumstances, the solutions are entire functions and the spectrum can only be discrete. It is easily proved directly from the governing Eq. (17) above that there can be no linear instability and all modes must be damped. This is due to the physical fact that the equilibrium has no free energy to drive instabilities (ideal or otherwise) in this case. We note too striking formal similarities between the above equation and the Orr-Sommerfeld equation of parallel flows in hydrodynamics (cf. Drazin and Reid [14]).

We next turn to an analytic theory of the spectral problem formulated above. It is useful to bear in mind that physically interesting limits are $\delta_e < 1$ and $\epsilon \ll 1$. We first neglect ϵ (electron viscosity) entirely and consider the outer limiting problem

$$\frac{d}{dx}(x\frac{df}{dx}) \quad = \quad \frac{\omega^2}{4}\frac{f}{(\omega^2\delta_e^2 - x)} \tag{19}$$

subject to the boundary conditions that f be regular at $x = 0$ and $df/dx = 0$ at $x = 1$.

It is worth remarking at this point that ideal MHD corresponds to letting $\delta_e \to 0$ in the above equation. Using the substitution $u = \ln x$ we can easily solve the equation. The boundary condition at $x = 1$ can be satisfied by taking $f = \cos(\frac{\omega}{2} \ln x)$ for arbitrary real ω. The origin $x = 0$ is a regular singular point which is in general a branch point for both linearly independent solutions. Furthermore, neither the current density nor the velocity is square integrable over the interval [0,1]. Hence the solutions belong to a continuous spectrum, namely the Alfvén continuum. Craig and McClymont resolved the singularity at the origin and obtained discrete, damped eigensolutions within the resistive MHD ($1/S \neq 0$) model. This model is not valid whenever $c/\omega_{pe} > L_\eta$, i.e. whenever, $\delta_e^2 > 1/S$. Our primary interest is in the latter situation, although we will make contact with the solutions of Craig and McClymont.

Neglecting resistive effects for the moment, we obtain the following key theorems relating to Eq. (19):

Theorem 1

Every non-trivial solution of the above boundary value problem must have $\mathrm{Im}(\omega^2\delta_e^2) = 0$.

Proof: Assume, to the contrary, $\mathrm{Im}(\omega^2\delta_e^2) \neq 0$. The equation is regular in the interval (0,1), except at the regular singular point $x = 0$. We may therefore multiply the equation by f^* (the complex conjugate of f) and integrate over (0,1), since both f and f^* must be regular throughout the interval. Similarly we multiply the equation satisfied by f^* by f and integrate. Thus we obtain the relations

$$-\int_0^1 x \frac{df}{dx}\frac{df^*}{dx} dx = \frac{\omega^2}{4} \int_0^1 \frac{ff^*}{(\omega^2\delta_e^2 - x)} dx \quad (20)$$

$$-\int_0^1 x \frac{df}{dx}\frac{df^*}{dx} dx = \frac{(\omega^*)^2}{4} \int_0^1 \frac{ff^*}{((\omega^*)^2\delta_e^2 - x)} dx \quad (21)$$

The right hand side of Eq. (20) can be written as $\frac{1}{4\delta_e^2}\left[\int_0^1 |f|^2 dx + \int_0^1 \frac{|f|^2 x dx}{(\omega^2\delta_e^2 - x)}\right]$. Using a similar transformation on the right hand side of Eq. (21) and subtracting the second equation from the first, we obtain the identity

$$0 = \mathrm{Im}(\omega^2\delta_e^2) \int_0^1 \frac{|f|^2 x dx}{|\omega^2\delta_e^2 - x|^2}. \quad (22)$$

Since the integral is positive definite, the result follows.

Theorem 2

A necessary condition for Eq. (19) to have a non-trivial solution is for the real number $\omega^2\delta_e^2 = \alpha^2$ to lie in the interval $0 < \alpha^2 < 1$.

Proof: By the previous result, ω^2 and hence α^2 must be real. Since the boundary conditions are real, we consider the equation when α^2 lies outside the interval [0,1]. Without loss of generality we can choose f itself to be a real function in this case. It follows that Eq. (20) is valid. We see at once that it leads to a contradiction if $\alpha^2 > 1$ (the right hand side will then be positive although the left hand side is negative) and also if $\alpha^2 < 0$, when, again the right hand side will be positive. The theorem follows.

This argument establishes that we must have a singular eigenvalue problem [9] with $\alpha^2 - x$ vanishing somewhere in the solution domain. We next prove that if this problem has a non-trivial solution, it must belong to the continuous spectrum [9].

Theorem 3

If the stated problem has a non-trivial eigensolution, both f and df/dx must be square-integrable over [0,1]. Thus the magnetic energy of the perturbation is bounded. However, the velocity perturbation and the current density perturbation are not square integrable and hence the eigensolutions must belong to the continuous spectrum, with eigenvalues satisfying $0 < \alpha^2 < 1$.

Proof: We have already seen that α^2 belongs to the interval. Using Frobenius theory (cf. Olver [15], Goursat [16]) it can be shown that the equation has two linearly independent solutions at the regular singular point $x = \alpha^2$. These solutions, which we denote by $\phi_r(z, \alpha^2)$, $\phi_s(z, \alpha^2)$ (where $z \equiv x - \alpha^2$, not to be confused with the usual cylindrical longitudinal coordinate which has no relevance in the present section), have the following well-known properties (Goursat[16]): $\phi_r(z, \alpha^2)$ is analytic at the singular point $z = 0$ and vanishes there. Furthermore, it can be normalized to have unit slope at this point. Its power series converges for $|z| < \alpha^2$ (i.e. as far as the nearest singularity, at $z = -\alpha^2$). It may be continued analytically everywhere in the finite complex z plane and represents a single-valued function in the cut plane, where the cut extends from $-\infty$ to $z = -\alpha^2$, along the negative real axis (it has a branch point at infinity and at $z = -\alpha^2$).

The solution that is singular at $z = 0$, ϕ_s, can be chosen to be unity at this point and has an expansion of the form $\phi_s = \psi_r(z, \alpha^2) + \phi_r(z, \alpha^2)\log(z)$, where $\psi_r(z, \alpha^2)$ is another analytic function. For real values of z, we may take $\phi_s = \psi_r(z, \alpha^2) + \phi_r(z, \alpha^2) \ln |z|$ on either side of $z = 0$. It can, of course be continued all the way to $z = 1 - \alpha^2$. As we shall discuss later, these functions can be expressed explicitly in terms of appropriate solutions of the hypergeometric equation (cf. Olver [15], Goursat[16]). They can also be readily computed numerically. It is clear from this discussion that f must be expressible (for real x) in terms of $\phi_{r,s}(z, \alpha^2)$. It is plain that both f and df/dx are square integrable (over $x \in [0, 1]$): this indicates that the magnetic energy is finite. Note however that the current density is proportional to $\frac{d}{dx}(x\frac{df}{dx}) \propto 1/(\alpha^2 - x)$. It is therefore not square integrable in the interval [0,1]. Since the perturbed velocity in the neighborhood of $x = \alpha^2$ is proportional to the perturbed current density [cf. Eq. (8)], we see that the eigensolutions cannot have finite kinetic energy. Hence all the acceptable eigensolutions must belong to the continuous spectrum [9]. We shall proceed to the actual construction of the continuum eigensolutions in terms of ϕ_r, ϕ_s.

Theorem 4

Let $u = x/\alpha^2$ be a real variable in $(0, \infty)$. The singular eigenvalue problem defined by,

$$\frac{d}{du}(u\frac{df}{du}) = (\frac{\alpha^2}{4\delta_e^2})\frac{f}{(1-u)}, \tag{23}$$

for real $0 < \alpha^2 < 1$, with f regular at $u = 0$ and $df/du = 0$ at $u = 1/\alpha^2$, has a class of continuum modes labelled by α^2 such that f can be chosen to be unity at the interior regular singular point $u = 1$.

Proof: Note that the eigenvalue α^2 occurs now in both the equation and the right hand boundary condition. However, the three regular singularities in the equation are now at the fixed points $u = 0, 1, \infty$.

We demonstrate the analytic construction (in principle) of the above class of solutions. In the case of continua, the eigenvalue is not determined by the boundary data, but may be chosen in advance as any point of the continuous spectrum (in this case, $0 < \alpha^2 < 1$). Hence let us start with a specific value α^2 in this interval and construct the eigenfunction $f_{\alpha^2}(u)$ satisfying Eq. (23) and the conditions of the theorem.

We denote by $\phi_{r,\alpha^2}(u)$ the solution of the second order ordinary differential equation Eq. (23) which is analytic at the regular singular point $u = 1$. We know from general theory that it exists and has a zero there. We may choose it to have unit slope at this point. We denote the second, linearly independent solution of the equation at $u = 1$ by $\phi_{s,\alpha^2}(u)$. As stated earlier, we know from the general theory that this function can be chosen to be unity at $u = 1$ and has a logarithmic singularity in its derivative. We will assume (again from the general theory of linear differential equations [16]) that these two linearly independent solutions can be extended all the way to $u = 1$ and to $u = 1/\alpha^2$.

For $u < 1$, the general solution we seek will be written as $f^-_{\alpha^2}(u) \equiv \phi_{s,\alpha^2}(u) + A^- \phi_{r,\alpha^2}(u)$. Since $u = 0$ is also a regular singularity of the equation with only one of the two solutions there being analytic (the other has a logarithmic branch point), we can find A^- uniquely by requiring the linear combination $f^-_{\alpha^2}(u)$ to be analytic at $u = 0$. Thus, $f^-_{\alpha^2}(u)$ is a solution of Eq. (23) (in [0,1)), which is analytic at $u = 0$ and has unit amplitude at $u = 1$. In the same manner, we set, for $u > 1$, $f^+_{\alpha^2}(u) \equiv \phi_{s,\alpha^2}(u) + A^+ \phi_{r,\alpha^2}(u)$. Here, the appropriate continuations of the functions to the domain are used. The constant A^+ must now be determined from the boundary condition. Since we must have $df^+_{\alpha^2}/du = 0$ at $u = 1/\alpha^2$, A^+ is generally determined by the equation

$$A^+ = -\left[\frac{\phi'_{s,\alpha^2}(1/\alpha^2)}{\phi'_{r,\alpha^2}(1/\alpha^2)}\right], \tag{24}$$

where the operator d/du is denoted by a prime.

In general, $\phi'_{r,\alpha^2}(1/\alpha^2)$ will not vanish and the above formulae complete the construction of the continuum eigenmode corresponding to α^2 in the entire interval [0,1]. This implies generically that the perturbed magnetic field (which is purely azimuthal in this $m = 0$ case) will be proportional to f'_{α^2}. It is zero at the origin, but tends to infinity logarithmically as $u \to \pm 1$. Note also that typically we will have $A^+ - A^- \neq 0$. In this event there will be in addition a finite discontinuity, corresponding to a current sheet. The first type of logarithmic singularity has an odd parity current sheet whilst the second type is a delta-function or even parity sheet. Note however that the potential itself (proportional to f_{α^2}) is continuous at the interior singularity. At $u = 1/\alpha^2$ the magnetic field vanishes, by construction.

We next consider the exceptional case mentioned above: values of α^2 for which $\phi'_{r,\alpha^2}(1/\alpha^2) = 0$. In this case, the continuum eigensolution has the following structure: $f^-_{\alpha^2} \equiv 0; f^+_{\alpha^2} \equiv \phi_{r,\alpha^2}(u)$ in $[1, 1/\alpha^2]$. According to this solution which fits all the requirements (except that the potential now vanishes at the singular point), the perturbed magnetic field is excluded (rather like the Meissner effect in superconductors) within the singularity and there is no logarithmic infinity. Instead, there is a purely symmetrical current sheet at the singular point and the magnetic field has a finite discontinuity at $u = 1$ (the eigenfunction has unit slope in the limit $u \to 1^+$) .

It is useful to give an equivalent and more physically appealing form of the solution. Note that $f^-_{\alpha^2}$ constructed above is analytic at $u = 0$ and can be continued past its logarithmic branch point at $u = 1$ all the way up to $u = 1/\alpha^2$. It follows that $f_{\alpha^2}(u)$ can then be written over the whole domain in the equivalent form,

$$f_{\alpha^2}(u) = f^-_{\alpha^2}(u) + (A^+ - A^-)H(u-1)\phi_{r,\alpha^2}(u), \tag{25}$$

where $H(u-1)$ is the standard Heaviside function. Substitution in Eq. (23) yields, upon making use of the properties of $\phi_{r,s;\alpha^2}$ and the well-known Dirac delta-function identities $x\delta(x) = 0$, $dH(x)/dx = \delta(x)$, the *inhomogeneous* differential equation

$$\frac{d}{du}(u\frac{df}{du}) = (\frac{\alpha^2}{4\delta_e^2})\left[\frac{f}{(1-u)}\right] + \Delta'\delta(u-1) \tag{26}$$

$$\Delta'(\alpha^2, \delta_e^2) = A^+ - A^- \tag{27}$$

We convert this equation into an equivalent integral equation, which is very convenient for numerical calculations. Thus we formally integrate Eq. (26) from $u = 0$, making use of the regularity of the solution at this point, obtaining:

$$\begin{aligned} u\frac{df}{du} &= (\frac{\alpha^2}{4\delta_e^2})\mathrm{P}\int_0^u \frac{f\,dt}{(1-t)} + \Delta' H(u-1), \\ &= (\frac{\alpha^2}{4\delta_e^2})\left[\int_0^u (\frac{f-1}{1-t})dt - \ln|u-1|\right] + \Delta' H(u-1), \end{aligned} \tag{28}$$

where we have a principal value integral at $u = 1$. We have then used the fact that $f = 1$ at $u = 1$ to convert the principal value integral into a convergent integral using standard properties of the logarithmic function. In this approach α^2 is supposed known and Δ' is the eigenvalue to be determined by the dispersion relation which f and Δ' must satisfy:

$$(\frac{\alpha^2}{4\delta_e^2})\left[\int_0^{1/\alpha^2}(\frac{f-1}{t-1})dt+\ln|\frac{1}{\alpha^2}-1|\right] \;=\; \Delta'. \tag{29}$$

These equations clearly exhibit the nature of the singularity at $u=1$ in the magnetic field. The logarithm corresponds to the odd parity current sheet whilst the Heaviside function describes the even parity current sheet.

Equation (28) can be integrated with respect to u about $u=1$. The resulting integral equation can be solved simultaneously with the linear constraint given by Eq. (29). This can be done numerically by iteration, starting with any reasonable initial f. As previously stated, in addition to the above eigensolutions, a different class of eigensolutions with the property that $f=0$ at $u=1$ can also be constructed by the integral equation technique. It is easily seen that if f is continuous everywhere and vanishes at $u=1$, we must have $f\equiv 0$ for $u<1$ in order to enforce the analyticity of the solution near the regular singularity at the origin. We can integrate Eq. (23) once and write (for $u\geq 1$)

$$u\frac{df}{du} \;=\; (\frac{\alpha^2}{4\delta_e^2})\int_1^u\frac{f(t)dt}{1-t}+1, \tag{30}$$

where we have made use of the fact that f vanishes at $u=1$ and its derivative is chosen to be unity. In this exceptional case, α^2 has to be determined: the dispersion relation for α^2 is obtained from

$$(\frac{\alpha^2}{4\delta_e^2})\int_1^{1/\alpha^2}\frac{f(t)dt}{t-1} \;=\; 1. \tag{31}$$

As before, Eqs. (30,31) can be solved by iteration. We have already noted that this type of solution must be written as $f_{\alpha^2}(u)=\phi_{r,\alpha^2}(u)H(u-1)$. It follows that the jump in the derivative at the singular point, $\Delta'=1$, and Eq. (31) follows from Eq. (26).

The results proved establish that the introduction of electron inertia to ideal MHD reduces the ideal MHD continuum ($0<\omega^2<\infty$) to a band-limited continuum, $0<\omega^2\delta_e^2<1$. In other words, electron inertia does not introduce damping or growth or discrete modes. It does, however, reduce the extent of the ideal MHD continuum. This is a remarkable fact in itself, showing that the interior singularity of the eigenvalue equations in the ideal case is merely altered, but not removed or resolved in electron MHD (ie, with electron inertia). As will be discussed later, the continuum eigenmodes in this case can be exactly expressed in terms of the solutions of the hypergeometric equation [15, 16].

4 Spectral Analysis Part II: $S < \infty, \epsilon \to 0$

We next consider the general inviscid case when S is finite but ϵ is still neglected. It is convenient to go back to the form of the problem suggested by Eq. (14). When $m = 0, \epsilon = 0$, we find that Eq. (18) may be written as

$$\begin{aligned} \frac{d}{dx}(x\frac{df}{dx}) &= \frac{\omega^2}{4}\frac{f}{(\alpha^2 - x)}, && (32)\\ \alpha^2 &= \omega^2\delta_e^2 + i\frac{\omega}{S}, && (33) \end{aligned}$$

where $0 \leq x \leq 1$ and the usual boundary conditions apply. We observe that unlike Eq. (19), we no longer have a Hermitian operator and the eigenvalues must therefore lie in the lower half of the complex ω-plane (since we know that the modes can only be damped).

It is useful to introduce the real frequency ω_0 and the decay rate γ (we will demonstrate that the latter must be positive), so that $\omega \equiv \omega_0 - i\gamma$. We also set $\alpha^2 \equiv a + ib$. It follows from simple algebra that

$$\begin{aligned} |\omega|^2 &= \omega_0^2 + \gamma^2 && (34)\\ a &= \mathrm{Re}(\alpha^2) \\ &= |\omega|^2\delta_e^2 + 2\gamma\delta_e^2(\frac{1}{2S\delta_e^2} - \gamma) && (35)\\ b &= \mathrm{Im}(\alpha^2) \\ &= 2\omega_0\delta_e^2(\frac{1}{2S\delta_e^2} - \gamma) && (36)\\ \omega^2 &= (|\omega|^2 - 2\gamma^2) - 2i\omega_0\gamma. && (37) \end{aligned}$$

The structure of the spectrum and the nature of the eigenmodes of Eq. (32) are elucidated by the following three theorems. It is convenient to treat the case of purely damped modes with real frequency, $\omega_0 = 0$ separately.

Theorem 5 (Discrete spectra with $\omega_0 \neq 0$):

If $\mathrm{Im}(\alpha^2) \neq 0$, it must be positive for $\omega_0 > 0$. Furthermore, $\mathrm{Re}(\alpha^2) = |\omega|^2\delta_e^2 + 2\gamma\delta_e^2(\frac{1}{2S\delta_e^2} - \gamma) < 1$. This component of the spectrum consists (for $\delta_e \neq 0$) of a finite number of discrete eigenvalues and damped eigenmodes, with damping rate γ satisfying $1/(2S\delta_e^2) > \gamma > 0$. The eigenvalues are determined by solving the transcendental dispersion equation involving the Gaussian hypergeometric function, $F(1 - i\frac{\omega}{2}, 1 + i\frac{\omega}{2}, 2; 1/\alpha^2) = 0$ (where α^2 is related to ω through Eq. (33)). The corresponding eigenfunctions have finite kinetic and magnetic field energies and have no singularities in $0 < x < 1$. If the dispersion relation has no solutions, the discrete spectrum does not exist in this case.

Proof:

When $\text{Im}(\alpha^2) = b \neq 0$, it is clear that the equation has no interior singularity. This immediately shows that not only the field but also the current density and velocity must be continuous over the whole solution domain (assuming that the eigenvalue problem has a nontrivial solution). Thus they must be square integrable and therefore belong to the discrete spectrum. It is also clear that ω cannot be zero since then the only solution of the eigenvalue problem is the trivial solution, $f = const.$ We can readily derive the following simple identities along the lines of earlier results (cf. Eqs. (20,21)):

$$-\frac{1}{\omega^2}\int_0^1 x|\frac{df}{dx}|^2 dx = \frac{1}{4}\int_0^1 \frac{|f|^2}{(\alpha^2 - x)} dx \quad (38)$$

$$-\frac{1}{(\omega^*)^2}\int_0^1 x|\frac{df}{dx}|^2 dx = \frac{1}{4}\int_0^1 \frac{|f|^2}{((\alpha^*)^2 - x)} dx \quad (39)$$

Let us consider the difference of these equations:

$$\left[(\omega^2) - (\omega^*)^2)\right]\int_0^1 x|\frac{df}{dx}|^2 dx = \frac{|\omega|^4}{4}\int_0^1 |f|^2\left[\frac{1}{\alpha^2 - x} - \text{c.c.}\right] dx, \quad (40)$$

where c.c. denotes complex conjugate. This relation yields certain key inequalities when simplified to

$$2\omega_0\gamma\int_0^1 x|\frac{df}{dx}|^2 dx = \frac{b|\omega|^4}{4}\int_0^1 \frac{|f|^2 dx}{|\alpha^2 - x|^2}. \quad (41)$$

Since $\omega_0 \neq 0$ by assumption, it can be assumed, without loss of generality, to be positive. Upon using Eq. (36), Eq. (41) immediately leads to

$$\gamma\left[\int_0^1 x|\frac{df}{dx}|^2 dx + \frac{\delta_e^2|\omega|^4}{4}\int_0^1 \frac{|f|^2 dx}{|\alpha^2 - x|^2}\right] = (\frac{1}{2S\delta_e^2})\left[\frac{\delta_e^2|\omega|^4}{4}\int_0^1 \frac{|f|^2 dx}{|\alpha^2 - x|^2}\right]. \quad (42)$$

This proves that

$$\frac{1}{2S\delta_e^2} > \gamma > 0. \quad (43)$$

From this we deduce the main results,

$$\begin{aligned}\text{Im}(\alpha^2) &= 2\omega_0\delta_e^2(\frac{1}{2S\delta_e^2} - \gamma)\\ &> 0 \quad (44)\\ \text{Re}(\alpha^2) &= |\omega|^2\delta_e^2 + 2\gamma\delta_e^2(\frac{1}{2S\delta_e^2} - \gamma),\\ &> |\omega|^2\delta_e^2 \quad (45)\end{aligned}$$

It is also easily established that $\text{Re}(\alpha^2) < |\omega|^2\delta_e^2 + \frac{1}{8S^2\delta_e^2}$. Adding Eqs.(38,39), and clearing fractions, we obtain,

$$-\left[\text{Re}(\omega^2)\right]\int_0^1 x|\frac{df}{dx}|^2 dx = \frac{|\omega|^4}{4}\int_0^1 |f|^2 \left[\frac{\text{Re}(\alpha^2) - x}{|\alpha^2 - x|^2}\right] dx$$

If $\text{Re}(\omega^2) = \omega_0^2 - \gamma^2$ is positive, the inequality $\text{Re}(\alpha^2) < 1$ is necessary to avoid a contradiction. If on the other hand $\text{Re}(\omega^2) < 0$, $\text{Re}(\alpha^2) < \gamma/S < 1/(2S^2\delta_e^2) < 1$, as assumed. Hence we have shown that $\text{Re}(\alpha^2) = a < 1$ in all cases.

Multiplying Eqs. (38,39) by ω^2 and its complex conjugate respectively and subtracting, we also obtain,

$$\frac{1}{4}\int_0^1 |f|^2 dx \left[\frac{\omega^2}{\alpha^2 - x} - \text{c.c}\right] = 0, \text{viz.},$$

$$\gamma\int_0^1 \frac{x|f|^2 dx}{|\alpha^2 - x|^2} = \frac{(|\omega|^2\delta_e^2)}{2S\delta_e^2}\int_0^1 \frac{|f|^2 dx}{|\alpha^2 - x|^2}. \tag{46}$$

These results imply that,

$$\gamma > \frac{(|\omega|^2\delta_e^2)}{2S\delta_e^2}, \tag{47}$$

somewhat strengthening Eq. (43). The latter now results in,

$$1 > |\omega|^2\delta_e^2 > 0. \tag{48}$$

To summarize, we have shown that if $\text{Im}(\alpha^2) \neq 0$ (and $\omega_0 > 0$), the equation has no singularity in the interior of the solution domain and we must have $\text{Im}(\alpha^2) > 0$, $|\omega|^2\delta_e^2 < 2\gamma S\delta_e^2 < 1$. We can readily convert the problem into equivalent (non-singular) integral equations. We now demonstrate that the conditions derived are also sufficient for the existence of eigensolutions.

The proof hinges on a transformation of Eq. (32) into the hypergeometric equation (cf. Olver [15], Goursat [16]). Setting $u = x/\alpha^2$ we obtain

$$u(1-u)\frac{d^2f}{du^2} + (1-u)\frac{df}{du} - \frac{\omega^2}{4}f = 0. \tag{49}$$

This is a particular form of the hypergeometric equation

$$u(1-u)\frac{d^2f}{du^2} + [c - (1+a+b)u]\frac{df}{du} - abf = 0,$$

with $a = -\frac{i\omega}{2}$, $b = \frac{i\omega}{2}$, $c = 1$. For these values, it is well known from standard theory [15] that the Gaussian hypergeometric function $F(a, b; c; u)$ is an analytic solution at $u = 0$.

This function has the convergent power series representation

$$F(a,b;c;u) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{u^n}{n!}, \tag{50}$$

valid in $|u| < 1$. Evidently, $F(a,b;c;0) = 1$. We also have $\text{Re}(c-a-b) = 1 > 0$. It follows that $F(a,b;c;1) = \frac{1}{\Gamma(1-a)\Gamma(1-b)}$.

Olver[15] also shows that F has an analytic continuation to the entire cut u-plane (cut running from $u = 1$ to infinity along the positive real u-axis), given by the integral,

$$F(a,b;c;u) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tu)^{-a}dt, \tag{51}$$

which is applicable for arbitrary u provided that $\text{Re}(a) < 1$ and $\text{Re}(c) > \text{Re}(b) > 0$. We can use this result to derive a dispersion relation for the discrete eigenvalues of Eq. (32). The boundary condition that f is analytic at $r = 0$ is automatically satisfied by the f given by Eq. (51) with the parameters as obtained above. At the other boundary $r = 1$, $df/dr = 0$ requires that $dF/dz = 0$. Using the differentiation formula [17]

$$\frac{d}{dz}F(a,b;c;z) = \frac{ab}{c}F(a+1,b+1;c+1;z), \tag{52}$$

we deduce that $f' = 0$ is satisfied if

$$F\left(1-\frac{i\omega}{2}, 1+\frac{i\omega}{2}; 2; \frac{1}{\alpha^2}\right) = 0. \tag{53}$$

In view of Eq. (51), bearing in mind that $\Gamma(2) = 1$ and $\Gamma(1 - i\omega/2)$, $\Gamma(2 + i\omega/2)$ are guaranteed to be finite for $\omega_0 \neq 0$, we infer that Eq. (52) is equivalent to

$$\int_0^1 t^{i\omega/2}(1-t)^{-i\omega/2}(1-t/\alpha^2)^{-1+i\omega/2}dt = 0. \tag{54}$$

It is straightforward to verify that the integral in this expression satisfies the conditions for the validity of analytic continuation provided that $0 < \gamma/2 < 1$. Equation (54) constitutes a dispersion relation for the discrete eigenvalues ω; having determined these, one can then construct the corresponding eigenfunction using

$$f = \frac{\Gamma(1)}{\Gamma(i\omega/2)\Gamma(1-i\omega/2)}\int_0^1 t^{i\omega/2-1}(1-t)^{-i\omega/2}(1-tr^2/\alpha^2)^{i\omega/2}dt. \tag{55}$$

As stated in the theorem, the dispersion relation for the discrete eigenvalues coincides with the Craig and McClymont eigensolutions when $\delta_e = 0$.

Theorem 6 (Continuous spectrum with $\omega_0 \neq 0$):

For these modes, it is necessary that $\text{Im}(\alpha^2) = 0$. If S, δ_e satisfy the criterion, $S\delta_e > 1/2$, a continuous spectrum always exists (for $\omega_0 \neq 0$). There is then an interior singularity at $x = \alpha^2$ where the eigenfunction f is continuous, together with a square integrable (but discontinuous) first derivative. It may be chosen to have unit amplitude at the singularity. However, the current density and the velocity are not square integrable and generally even and odd parity current sheets are simultaneously present. The damping rate of the modes with non-zero real frequency is independent of α^2 and is given by $\gamma = 1/(2S\delta_e^2)$. The real frequency, ω_0 satisfies the inequality, $0 < \omega_0^2\delta_e^2 < 1 - \frac{1}{4S^2\delta_e^2}$. The continuum eigenmodes with nonvanishing real frequencies can always be chosen to have unit amplitude at the interior singular point.

Proof:

We now consider the possibility that $\text{Im}(\alpha^2) = 0$ and assume that the real frequency $\omega_0 \neq 0$. From the preceding arguments, we have immediately, $\gamma = 1/(2S\delta_e^2)$ in this case. Defining an angle θ by the relation $\sin\theta = \gamma/|\omega|$, we see that $\omega = |\omega| \exp(-i\theta)$. It is convenient to write $|\omega| = \Omega = (\omega_0^2 + \frac{1}{4S^2\delta_e^4})^{1/2}$. Evidently $\alpha^2 = \Omega^2\delta_e^2, \omega_0 = \Omega\cos\theta$ in this case.

It is convenient to write the eigenvalue equation [Eq. (32)] in the form

$$(\alpha^2 - x)\frac{d}{dx}(x\frac{df}{dx}) = \left[\frac{\alpha^2\exp(-2i\theta)}{4\delta_e^2}\right] f, \tag{56}$$

with the boundary conditions that f be regular at $x = 0$ and $\frac{df}{dx} = 0$ at $x = 1$. For a given value of the parameter α^2, θ is a *definite function* of α, S and δ_e via $\gamma[= 1/(2S\delta_e^2)]$ through its defining relation $\sin\theta = 1/(2\Omega S\delta_e^2)$.

We can show immediately, using earlier arguments, that a nontrivial solution of the problem only exists if $0 < \alpha^2 < 1$. If α^2 lies outside this interval, the equation is non-singular and the integral relations lead to a contradiction. Thus we may assume without loss of generality that $\alpha^2 - x$ vanishes at a point in the unit interval. To proceed further we demonstrate some properties of solutions of Eq. (56).

Consider Eq. (56) in $0 \leq x < \infty$, where $0 < \alpha < 1$ and $0 < \theta < \pi/2$ are otherwise arbitrary real parameters and δ_e is a fixed real parameter (where $0 < \delta_e < 1$). We can construct a solution of this equation $\Phi_{r,0}(\alpha^2, \theta; x)$ (the suffix r stands for "regular/single-valued analytic" function of the complex variable x) with the following properties:

a). $\Phi_{r,0}(\alpha^2, \theta; x)$ is analytic as a function of the complex variable x for $|x| < \alpha^2$ and is nonzero for $x = 0$.

b). $\Phi_{r,0}(\alpha^2, \theta; x)$ can be chosen to have the value unity at $x = \alpha^2$, where it has, as a function of the complex variable, $x - \alpha^2$, a logarithmic branch point and $d\Phi_{r,0}/dx \simeq \ln(|x - \alpha^2|)$. Furthermore, $\Phi_{r,0}(\alpha^2, \theta; x)$ has an analytic continuation (in the cut plane, with a cut running from $x = \alpha^2$ to ∞, but the principal branch is valid on the cut) beyond $x = \alpha^2$.

To prove these statements, we use the properties of the hypergeometric equation form of Eq. (56). Setting $u = x/\alpha^2$ we obtain

$$u(1-u)\frac{d^2f}{du^2} + (1-u)\frac{df}{du} - \frac{\alpha^2 \exp(-2i\theta)}{4\delta_e^2} f \;=\; 0. \tag{57}$$

This is a particular form of the hypergeometric equation with $a = -\frac{i\alpha}{2\delta_e}\exp(-i\theta)$, $b = \frac{i\alpha}{2\delta_e}\exp(-i\theta)$, $c = 1$. As stated earlier, the Gaussian hypergeometric function $F(a,b;c;u)$ is an analytic function at $u = 0$. It is also known [16, 17] that for the parameters used in our application

$$\begin{aligned} F(a,b;1;u) &= \frac{1}{\Gamma(1+a)\Gamma(1+b)} \\ &\quad - \frac{\sum_{n=0}^{\infty} W_n \left[Q_n - \log(1-\mathrm{u})\right](1-u)^{n+1}}{\Gamma(a)\Gamma(b)\Gamma(1+a)\Gamma(1+b)}, \\ W_n &= \frac{\Gamma(1+a+n)\Gamma(1+b+n)}{(1+n)!n!}, \end{aligned} \tag{58}$$

where the coefficients Q_n can be expressed in terms of logarithmic derivatives of the Gamma function [16, 17]. This formula is valid for $|u-1| < 1$ and $\arg|(1-u)| < \pi$ and a, b do not take zero or negative integer values. It shows that F clearly has a logarithmically infinite derivative at $u = 1$. We have already seen (cf. Eq. (51)) that that F has an analytic continuation to the entire cut u-plane (cut running from 1 to ∞) given by Euler's integral,

$$F(a,b;c;u) \;=\; \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tu)^{-a} dt, \tag{59}$$

which is applicable for arbitrary u provided that $\mathrm{Re}(a) = -\frac{\alpha}{2\delta_e}\sin\theta < 1$ and $Re(c) = 1 > Re(b) = \frac{\alpha}{2\delta_e}\sin\theta > 0$. Equation (59) holds in particular when u is real and greater than unity. By setting $\Phi_{r,0}(\alpha^2, \theta; x) \equiv F(a,b;1;u\alpha^2)\Gamma(1-a)\Gamma(1-b)$, we establish the stated result (a) above.

We next consider the solution of Eq. (56) which is analytic at $x = \alpha^2$. Consider Eq. (56) in $0 \le x < \infty$, where $0 < \alpha < 1$ and $0 < \theta < \pi/2$ are otherwise arbitrary real parameters and δ_e is a fixed real parameter (where $0 < \delta_e < 1$). We can construct a solution of this equation $\Phi_{r,1}(\alpha^2, \theta; x)$ with the following properties:

a) $\Phi_{r,1}(\alpha^2, \theta; x)$ is *analytic* as a function of the complex variable x for $|x - \alpha^2| < \alpha^2$ and vanishes at $x = \alpha^2$.

b) $\Phi_{r,1}(\alpha^2, \theta; x)$ can be chosen to have unit slope at $x = \alpha^2$. As a function of the complex variable, $x - \alpha^2$ $\Phi_{r,1}(\alpha^2, \theta; x)$ has an analytic continuation (in the cut plane, with a cut running from $-\infty$ to $x = \alpha^2$) for $x > \alpha^2$.

c) The derivative, $\frac{d\Phi_{r,1}}{dx} \neq 0$ for $x \ge \alpha^2$, for any permissible values of α^2, θ.

To prove these statements, we consider the transformed Eq. (57). From standard theory[15, 16, 17] we know that at $u = 1$ there exists an analytic solution $(u-1))F(1-a, 1-b; 2; 1-u) = \Phi_{r,1}(\alpha^2, \theta; u)$, which evidently vanishes there and has unit slope. This solution is expressible as a power series in $1-u$ with a radius of convergence unity. It can be analytically continued by expressing it in terms of the two linearly independent solutions at the regular singularity at ∞.

Thus, we need only prove c) in detail. Consider Eq. (57) written in the form

$$(1-u)\frac{d}{du}(u\frac{dF}{du}) = \left[\frac{\alpha^2 \exp(-2i\theta)}{4\delta_e^2}\right] F, \tag{60}$$

with $F \equiv \Phi_{r,1}(\alpha^2, \theta; u)$. Assume, contrary to hypothesis, that there is a $u_+ > 1$, such that the derivative, $\frac{d\Phi_{r,1}}{du}$ vanishes there. We multiply the equation by $F^*/(1-u)$ and integrate over $[1, u_+]$, bearing in mind the fact that this function is analytic at $u = 1$ and using the assumed condition at u_+. We obtain the integral identity (all integrals are convergent),

$$-\int_1^{u_+} u|\frac{dF}{du}|^2 du = \frac{\alpha^2 \exp(-2i\theta))}{4\delta_e^2} \int_1^{u_+} \frac{|F|^2}{1-u} du \tag{61}$$

Subtracting this equation from its complex conjugate, we derive the contradiction that a positive definite integral must vanish. Hence the result stated has been proved.

From this point on, we simply follow the procedure used in Theorem 3 and obtain the continuum eigensolutions when $\mathrm{Im}(\alpha^2) = 0$ and $\omega_0 \neq 0$. The following theorem summarizes the results, which follow *mutatis mutandis* from the arguments used in the proof of Theorem 3.

Theorem 7

1. If $0 < \alpha^2 < 1$, the eigensolution of Eq. (60) with unit amplitude at $u = 1$ belongs to the continuous spectrum and can be written in the form

   $$F(\alpha^2, \theta; u) = \Phi_{r,0}(\alpha^2, \theta; u) + \Delta'(\alpha^2, \theta, \delta_e^2)\Phi_{r,1}(\alpha^2, \theta; u), \tag{62}$$

   where Δ' is determined by the dispersion relation,

   $$\Delta' = -\frac{\Phi'_{r,0}(\alpha^2, \theta; 1/\alpha^2)}{\Phi'_{r,1}(\alpha^2, \theta; 1/\alpha^2)}. \tag{63}$$

   In Eq. (63) the denominator cannot vanish when $\omega_0 \neq 0$ (by the previous theorem).

2. Furthermore, F satisfies the integro-differential equations,

   $$u\frac{dF}{du} = (\frac{\alpha^2 \exp(-2i\theta)}{4\delta_e^2}) \left[\int_0^u (\frac{F-1}{1-t})dt - \ln|u-1|\right] + \Delta' H(u-1), \tag{64}$$

$$(\frac{\alpha^2 \exp(-2i\theta)}{4\delta_e^2}) \left[\int_0^{1/\alpha^2} (\frac{F-1}{t-1})dt + \ln|\frac{1}{\alpha^2} - 1|\right] = \Delta'. \quad (65)$$

3. There are no eigensolutions which vanish at $u = 1$. The damping rate γ of these eigensolutions is independent of α^2 and is given by $\gamma = 1/(2S\delta_e^2)$.

We take up now the special case of zero real frequency (ie, purely damped) modes. We know that they cannot occur if $1/S = 0$. The situation is different and more interesting when $1/S, \delta_e$ are both nonzero.

Theorem 8: Spectral properties of purely damped ($\omega_0 = 0$) eigenmodes

If $\omega_0 = 0$ In Eq. (32) we must have $\mathrm{Im}(\alpha^2) = 0$. Furthermore, $\mathrm{Re}(\alpha^2) = a = \gamma\delta_e^2(\frac{1}{S\delta_e^2} - \gamma)$ satisfies the inequalities $0 < \alpha^2 < \frac{1}{4S^2\delta_e^2}$. If $\frac{1}{4S^2\delta_e^2} < 1$, there is a purely continuous spectrum of damped modes with $0 < \gamma < \frac{1}{S\delta_e^2}$. If $\frac{1}{4S^2\delta_e^2} > 1$, there will be a continuum as well as a possibly non-empty discrete spectrum of damped modes.

Proof

It is obvious from Eq. (36) that in this case $\mathrm{Im}(\alpha^2) = 0$. From Eq. (35) it follows that $\alpha^2 = \gamma\delta_e^2[1/(S\delta_e^2) - \gamma]$. If $\alpha^2 < 0$, Eq. (32) can be easily seen to have no solution, using the argument of Theorem 2. It then follows that $0 < \gamma < 1/(S\delta_e^2)$. By maximising a over this range, we infer that $0 < \alpha^2 < 1/(4S^2\delta_e^2)$. Evidently, there are two possibilities. If $1/(4S^2\delta_e^2) < 1$, there will be a real interior singularity for each allowed value of γ and we can construct a continuum exactly as we did in Theorem 4. If $1/(4S^2\delta_e^2) > 1$, the continuum extends only for $0 < \alpha^2 < 1$. For $1 < \alpha^2 < 1/(4S^2\delta_e^2)$, there are no interior singularities and only discrete solutions can be expected. These may or may not exist depending upon the conditions. We indicate their construction.

Now suppose $1 < \alpha^2 \leq 1/(4S^2\delta_e^2)$. Equation (32) can be transformed into the hypergeometric equation by setting $u = x/\alpha^2$, $a = -\gamma/2$, $b = \gamma/2$, $c = 1$. The solution which is analytic at $x = 0$ is $F(a, b; 1; x/\alpha^2)$. We need to impose the boundary condition at $x = 1$. This, as discussed earlier, takes the form of the transcendental equation $F(1 - \gamma/2, 1 + \gamma/2; 2; 1/\alpha^2) = 0$, with $1 < \alpha^2 \leq 1/(4S^2\delta_e^2)$, and γ is a root of the quadratic equation $\alpha^2 = \gamma\delta_e^2[1/(S\delta_e^2) - \gamma]$. In this case the power series for the hypergeometric function can be used, since the argument $1/\alpha^2 < 1$, by assumption. Each root of the transcendental equation satisfying the constraints will contribute a discrete eigenmode. Note also that when $\alpha^2 > 1$, Eq. (32) *can* allow solutions since we have $-\gamma^2/4$ in the right hand side rather than $\omega^2/4$, which appears in Eq. (32) and, as shown in Theorem 2, disallows $\alpha^2 > 1$ solutions in the case of finite ω_0 and $S \to \infty$. The discrete $\omega_0 = 0$ modes, if they exist, are of little physical relevance, since they will be very strongly damped for realistic values of S and δ_e.

It is of interest to note that $1/(4S^2\delta_e^2) > 1$ is precisely the criterion found for the $\omega_0 \neq 0$ continuum to disappear. It is also of interest to note that these purely damped continua exist for every finite δ_e, so long as $1/S > 0$, however small.

This completes the determination of the spectra of this problem. In general the continuous and discrete spectra determined by the above theorems coexist. In the limit as $S \to \infty$ for fixed δ_e, the discrete spectrum disappears and the continuum merges smoothly with the ideal electron MHD continuum found in the previous section. As $\delta_e \to 0$ for fixed S, the continuum with $\omega_0 \neq 0$ disappears (when $\delta_e = \frac{1}{2S}$). The purely damped continuum remains until $\delta_e = 0$. In this limit, the discrete spectrum coincides with the Craig and McClymont discrete spectrum. If one takes the ordered limit $S \to \infty$ after $\delta_e \to 0$, one recovers the ideal MHD continuum.

We have established the spectral properties in the Case 1, when $m = k = 0$. We now indicate how the analysis extends to the situation when $m \neq 0$ and k is arbitrary. Equation (32) now becomes (NB Eq. (33) relating α^2 to ω still applies),

$$\frac{d}{dx}(x\frac{df}{dx}) - \frac{1}{4}(\frac{m^2}{x} + k^2)f = \frac{\omega^2}{4}\frac{f}{(\alpha^2 - x)} \tag{66}$$

The boundary conditions are: f must be regular at $x = 0$ and vanish at $x = 1$. It is elementary to verify that Theorems 1 and 2 are hardly modified, since the extra terms merely add a negative definite contribution to the left. Theorem 3 is also applicable since the points $x = 0, 1$ remain regular singular points of the equation. Theorem 4 will hold with only very minor changes brought about by $f = 0$ at $x = 1$. Note however that the integral equations have to be modified slightly to account for the different boundary condition at $x = 1$. In particular, the constraint given by Eq. (29) must be replaced by one which requires $f(1) = 0$. It is now clear that Theorem 5 also applies, *mutatis mutandis*. The eigenvalue equation is no longer the hypergeometric equation however, since it will have an irregular singular point at infinity. This hardly changes the general method of construction of the discrete and continuum eigenfunctions, apart from the inapplicability of the theory of the hypergeometric function and its analytical continuations. An equivalent theory can be easily constructed for Eq. (66).

5 Numerical Solutions in Inviscid Limit

5.1 Shooting method

We consider the $\epsilon = 0$, $m = 0$ case of the eigenvalue problem with boundary conditions $f' = 0$ at $r = 0$ and $r = 1$. Both discrete and continuum eigenvalues of Eq. (14) can be obtained using the shooting method; it is then straightforward to construct numerically the corresponding eigenfunctions. Having determined $f(r)$, it is instructive to compute the azimuthal field perturbation $\tilde{B}_\varphi \propto -f'$ and the longitudinal current perturbation $\tilde{j}_z \propto -f'' - f'/r$. The momentum equation [Eq. (8)] indicates that the latter quantity is related in a simple way to the fluid velocity and displacement.

Figure 1 shows $\tilde{B}_\varphi$ and $\tilde{j}_z$ eigenfunctions computed numerically for the case $S = 10^3$, $\delta_e = 0.01$. The mode shown in the upper two plots belongs to the discrete spectrum, while the mode in the lower plots lies in the continuum. In the discrete spectrum, both the real frequency and the damping rate increase with the number of radial nodes: the mode with $\omega_0 = 0.80$ in Fig. 1 has the lowest frequency and damping in the discrete spectrum. Qualitatively, this mode is similar to those found by Craig and McClymont [3] for $\delta_e = 0$. However, there are no continuum modes in the model considered by these authors, except in the ideal limit ($S \to \infty$). Figure 1 illustrates the important point that the continuum exists for finite values of S, provided that δ_e is also finite.

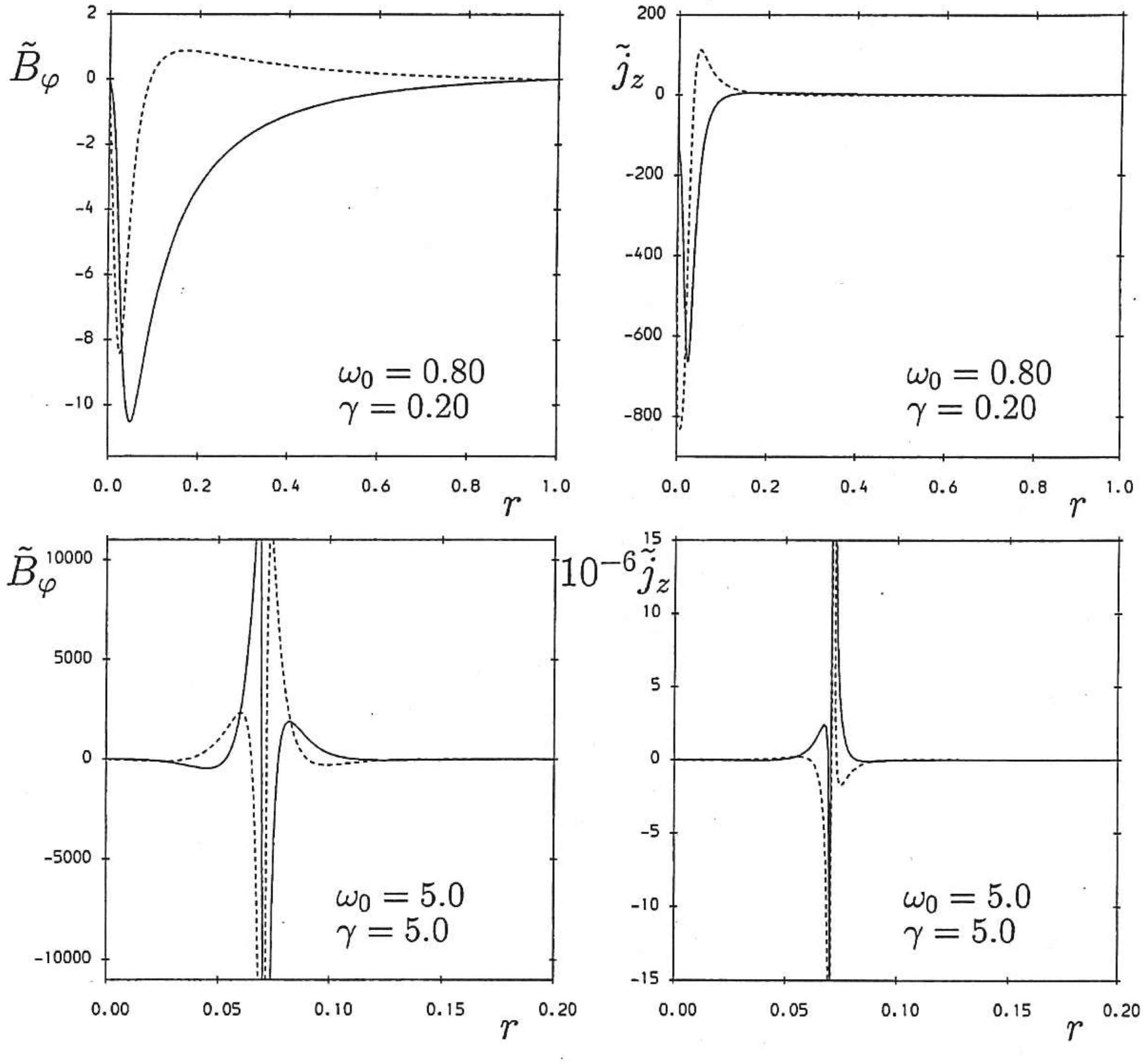

Figure 1: Real (solid curves) and imaginary (broken curves) parts of $\tilde{B}_\varphi$ (left plots) and $\tilde{j}_z$ (right plots) for $\epsilon = 0$, $m = 0$, $\delta_e = 0.01$ and $S = 10^3$. The upper plots show an example of a discrete eigenmode, while the lower plots show a continuum eigenmode.

Two characteristic dimensionless length scales appear in Eq. (14): $\omega_0\delta_e$ and $(\omega_0/S)^{1/2}$. For values of δ_e such that $\omega_0\delta_e < (\omega_0/S)^{1/2}$, we find modes that are similar to that shown in the upper plots of Fig. 1, insofar as the eigenfunctions are analytic. Keeping S fixed, however, we find that the eigenmodes become singular, and the eigenvalue spectrum continuous, when δ_e is increased to the point where the resistive length becomes comparable to the inertial length. To illustrate this change of character, it is useful to consider α^2, defined as before by

$$\alpha^2 \equiv \omega^2\delta_e^2 + i\omega/S. \tag{67}$$

In general, α^2 is complex. As we have seen earlier, the eigenvalue problem becomes singular if $\text{Im}(\alpha^2) = 0$. Figure 2 shows the imaginary part of α^2 for $S = 10^3$ and δ_e increasing from 0 to 0.05.

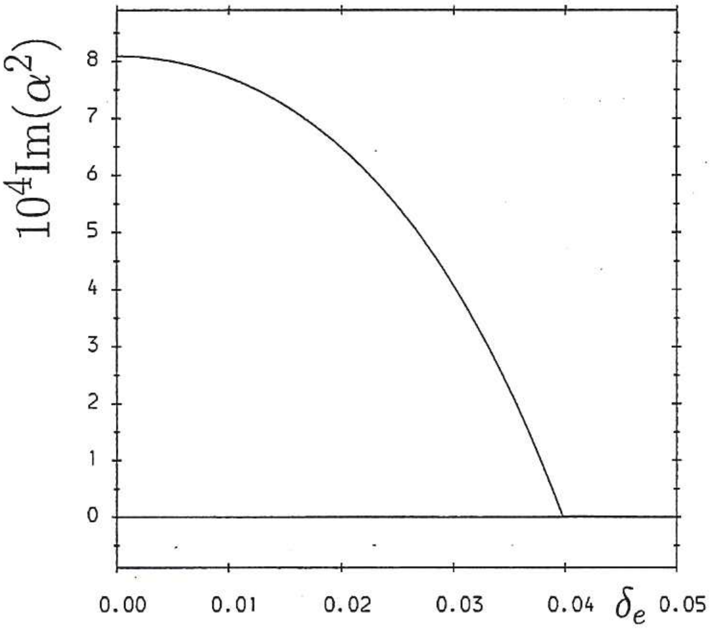

Figure 2: Imaginary part of $\alpha^2 \equiv \omega^2\delta_e^2 + i\omega/S$ versus δ_e for $\epsilon = 0$, $m = 0$ and $S = 10^3$.

The curve was obtained by computing ω_0 and γ in the limit $\delta_e = 0$ for the lowest frequency discrete mode, and tracking this mode as δ_e increases. We find that $\text{Im}(\alpha^2)$ approaches zero and then remains there: there is a discontinuity in the derivative of $\text{Im}(\alpha^2)$ with respect to δ_e. At this point, the discrete mode merges with the continuum. It is important to note, however, that the continuum exists for values of δ_e below that at which the curve in Fig. 2 crosses the δ_e axis: this is indicated in Fig. 2 by a solid line extending along the entire δ_e axis. The continuum mode shown in Fig. 1, for example, corresponds to the point $\delta_e = 0.01$, $\text{Im}(\alpha^2) = 0$ in Fig. 2.

As we have seen in the previous section, $\text{Im}(\alpha^2)$ vanishes if the damping rate γ satisfies

$$\gamma = \frac{1}{2S\delta_e^2} \tag{68}$$

This scaling is in marked contrast to the much weaker (logarithmic) dependence on S found in the limit $\delta_e \to 0$ by Craig and McClymont[3]:

$$\gamma \simeq \frac{\pi^2}{2(\ln S)^2}. \tag{69}$$

We have found numerically that when $\delta_e = 0$ and f' rather than f is set equal to zero at $r = 1$, the damping still varies as $(\ln S)^{-2}$, although its magnitude is slightly greater than that given by Eq. (69). For a given δ_e, the damping rate scales according to Eq. (68) when S is sufficiently large. Tokamak plasmas are characterised by very large Lundquist numbers, and so the damping of these continuum modes could be vanishingly small. In a real system, with a non–potential equilibrium field involving currents and/or energetic particles, it is possible that the modes could be driven unstable.

Figure 3 shows the relationship between the discrete and continuous eigenvalue spectra for one particular pair of values of S and δ (as in Fig. 1, 10^3 and 0.01 respectively). In this case there are ten discrete modes with damping rates lying below the continuum value given by Eq. (68). We showed earlier that no modes can exist with γ greater than $1/2S\delta_e^2$. The discrete modes are essentially unaffected by the existence of the continuum until they merge with it. In the model of Craig and McClymont, there is no upper limit to ω_0 or γ in the discrete spectrum. As δ_e is increased for fixed S, the continuum moves towards the real axis in the complex ω plane, annihilating as it does so the modes in the discrete spectrum. When the fundamental (lowest frequency) discrete mode merges with the continuum, the discrete spectrum ceases to exist.

To understand the nature of the singular eigenfunction solutions of Eq. (18), we consider the limit $S \to \infty, \epsilon \to 0$, i.e. $\alpha^2 = \omega^2\delta_e^2$. For $m = 0$ Eq. (19) is obtained. We have already discussed the nature of the solutions near the interior singularity. Since logarithmic singularities are integrable, the total magnetic field energy is finite. We have already discussed the different species of eigenmode solutions of Eq. (19)).

We have numerically obtained both classes of solution. Figure 4 shows $\tilde{B}_\varphi$ and $\tilde{j}_z$ for an eigenmode of the first type, with a logarithmic singularity in $\tilde{B}_\varphi$ at $r = \omega\delta_e$, and Fig. 5 shows a solution of the second type. In the latter case $\tilde{B}_\varphi$ is finite everywhere, but is discontinuous at $r = \omega\delta_e$. Since the perturbed current density is proportional to $-\nabla^2\tilde{\psi} \sim -(f'' + f'/r)$, there is a cylindrical current sheet at this radius, as shown in Fig. 5(b). Numerically, it is straightforward to find modes similar to that shown in Fig. 4 for finite values of S and α^2 real (i.e. the regime in which Eq. (32) is singular). In this case the damping rate is invariably equal to the value given by Eq. (68), as foreseen by the analysis of the preceding section.

5.2 Dispersion relation for discrete spectrum

Craig and McClymont[3] noted that for the case of $m = 0, k = 0,\ \delta_e = 0$ Eq. (14) reduces to the hypergeometric equation. We have seen that this remains true when

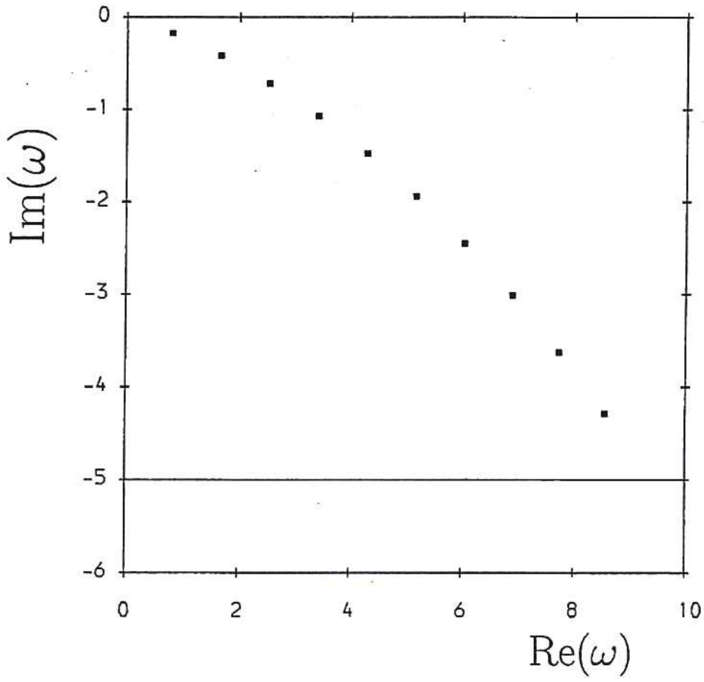

Figure 3: The discrete eigenvalue spectrum for $\epsilon = 0$, $m = 0$, $\delta_e = 0.01$ and $S = 10^3$. The solid line at $\text{Im}(\omega) = -5$ indicates the continuum.

δ_e is finite, and have discussed some properties of the solutions which are useful in constructing the continuum eigensolutions. We apply the hypergeometric function solutions to the discrete spectrum which exists when $\text{Im}(\alpha^2) > 0$.

The solutions of Eq. (56) which are analytic at $x = 0$ take the form (as already noted)

$$f = F\left(-\frac{i\omega}{2}, \frac{i\omega}{2}; 1; \frac{r^2}{\alpha^2}\right), \tag{70}$$

where $F(a, b; c; u)$ is the hypergeometric series, Eq. (50). This series has circle of convergence $|z| = 1$; since $|\alpha^2|$ is generally less than unity, the hypergeometric series cannot be used to represent eigenfunction solutions of Eq. (14) in the domain $|\alpha| < r \leq 1$. However, the hypergeometric function can be defined for $|z| > 1$ by analytical continuation, as in Eq. (51).

The integral in Eq. (51) gives the principal value of F provided that $0 < \gamma/2 < 1$; a particular form of the same integral appears in the dispersion relation for the discrete spectrum, Eq. (54). It is necessary to exercise care in solving this dispersion relation, since the integrand contains factors such as $\cos[(\omega/2)\ln t]$, which pose difficulties for numerical integration routines in the limit $t \to 0$. Singular points of this type occur at both end of the integration interval and, in the case of real α^2, at $t = \alpha$ (it should be noted, however, that the eigenvalues are not determined by Eq. (51) in this case, since they lie in the continuous spectrum). In the immediate vicinity of each of these

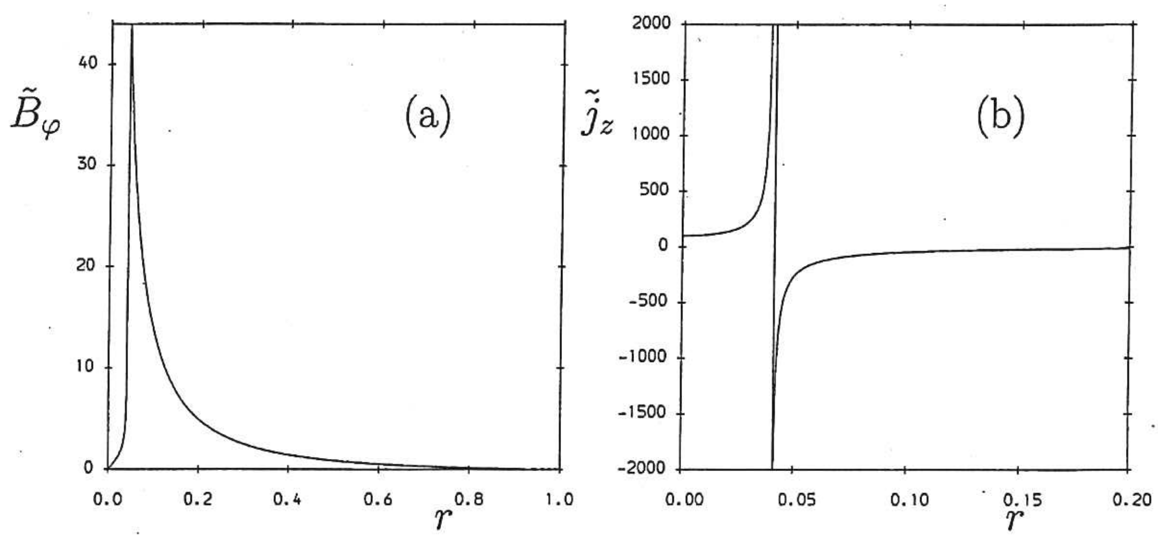

Figure 4: Example of a continuum eigenfunction in the limit $S \to \infty$. The parameters are $\epsilon = 0$, $m = 0$, $\delta_e = 0.1$, and the plotted quantities are (a) azimuthal field perturbation and (b) longitudinal current perturbation. The (real) eigenvalue is $\omega \simeq 0.40$.

points, only one of the three factors in the integrand varies significantly; the other two can be regarded as constant. Thus, one can evaluate contributions to the integral from neighbourhoods of the points $t = 0$, 1 and α analytically, and the remainder of the integral numerically.

Evaluating the integral in Eq. (54) in this way for particular values of S and δ_e, we find that it vanishes for complex frequencies ω lying very close to the discrete eigenvalues obtained using the shooting method. However, this method of eigenvalue determination fails when α^2 approaches the real line. The hypergeometric functions may also be used (with care) in the continuum case. However, the values are needed along the cut and the analytic continuation of the Gaussian hypergeometric series alone cannot give the answer. In particular the current sheets prevent simple application of tabulated functions. However, both the shooting method and the integral equation method are powerful enough to deal with the continuum case, as we demonstrated in the previous subsection.

6 Viscous Layer Theory in Limit $S \to \infty, \epsilon \ll 1$

We now treat the inner equations developed from Eq. (17). For simplicity of treatment we shall assume that $\omega\delta_e = \alpha < 1$ is a fixed parameter whilst $\epsilon \to 0$. Introducing

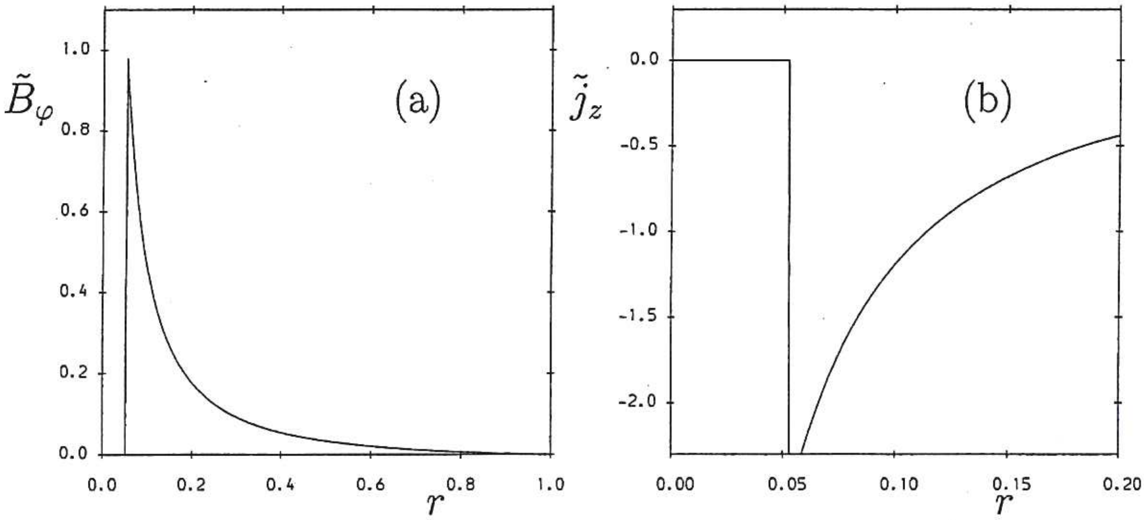

Figure 5: Continuum eigenfunction in the limit $S \to \infty$ with zero amplitude inside $r = \omega\delta_e$. There is a current sheet (more precisely, a current cylinder) at this point. As in Fig 4, $\epsilon = 0$, $m = 0$ and $\delta_e = 0.1$. The eigenvalue is $\omega_0 \simeq 0.52$ and the plotted quantities are (a) azimuthal field perturbation and (b) longitudinal current perturbation.

$x = r^2$, and considering $m = 0$, Eq. (17) can be written in the form

$$\frac{\alpha^2}{\delta_e^2} f + (x - \alpha^2) 4 D^2 f \quad = \quad -i\alpha\epsilon 16 D^4 f \tag{71}$$

Clearly close to the singularity, "dominant balance" suggests the inner layer scaling, $x = \alpha^2 + \alpha\epsilon^{1/3}\tilde{x}$, where $\tilde{x}$ is a "stretched" layer variable. Bearing in mind the fact that $f = 1$ at the singularity and is continuous, the eigenvalue equation for f can then be written as a pair of coupled second-order differential equations for f and the layer current-density, g:

$$\frac{d^2 f}{d\tilde{x}^2} \quad = \quad g \tag{72}$$

$$\tilde{x} g + \left(\frac{\alpha\epsilon^{1/3}}{4\delta_e^2}\right) f \quad = \quad -4i \frac{d^2 g}{d\tilde{x}^2} \tag{73}$$

In the absence of the viscous term, the current density clearly blows up near the interior singularity $x \simeq \alpha^2$ like $1/(\alpha^2 - x)$, when f itself is non-zero there. Let us first consider the case when $f \to 1$ as $x \to \alpha^2$. It is evident that in the layer, we may replace f by unity (in leading order of ϵ) and obtain, upon introducing the *complex variable*

$Z = (4\exp(-i\pi/2))^{1/3}\tilde{x}$, and $g = G(\frac{\alpha\epsilon^{1/3}}{4\delta_e^2})(4\exp(-i\pi/2))^{-1/3}$

$$\frac{d^2G}{dZ^2} = ZG + 1 \tag{74}$$

This is an inhomogeneous Airy equation [15], whose properties are well-known. We defer the detailed anlysis of the inner layer to a later work, but merely note here that the continuous spectra will be resoved into discrete eigenvalues in the lower half ω-plane by the viscous term. The damping rate is expected to scale like $\epsilon^{1/3}$. This resolution is analogous (both physically and mathematically) to that of the flow continuum by ordinary viscosity encountered in the much simpler case of the advection-diffusion equation [13].

7 Discussion and Conclusions

We have determined the spectrum of modes associated with a current–free magnetic X–point, taking into account resistivity and electron inertia in Ohm's law. In the limit of zero resistivity and viscosity, we have shown that the equation describing perturbations to the X–point equilibrium has two classes of singular eigenfunctions (continuuous spectra). For small finite values of resisitivity and zero viscosity, there are two cases. For fixed dimensionless skin depth δ_e, the spectrum has both discrete and continuous components when the Lundquist number S is sufficiently small. Remarkably, when the Lundquist number exceeds a critical threshold we again obtain a continuum, although the modes are weakly damped. We have also found a continuum and possibly discrete solutions under well-defined conditions of purely damped, zero frequency modes. Thus, in this problem, the singularity is not resolved by resistivity in the presence of electron inertia, if the skin depth exceeds the resistive length scale sufficiently. Electron viscosity is expected to resolve the singularities fully and introduce purely discrete spectra. We have indicated the key elements of a deeper analysis of the critical layers. A full treatment of the layer theory is left for future work.

These results indicate the existence of eigenmodes with frequencies typically in the Alfvén range, which could redistribute or accelerate energetic ions in their vicinity. They could in principle also be involved in nonlinear turbulent transport processes in the vicinity of X-point configurations. It should be stressed that the problem should be regarded as a paradigm for more realistic and complex field configurations with the same X-point topology. We have omitted several possibly important effects which may significantly modify the spectral properties. These include finite ion gyro radius, equilibrium currents, pressure gradients, field curvatures, compressibility, flows, and many kinetic effects. Some of these will introduce gaps and gap modes in the continuous spectra, and possibly new branches of the spectrum (e.g. drift Alfvén or trapped particle modes). An understanding of the linear eigenmode spectrum is of fundamental importance for nonlinear calculations of turbulence and spectral transfer processes.

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