United Kingdom Atomic Energy Authority

UKAEA FUS 528

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Toroidal and poloidal flows in single-fluid and two-fluid tokamak equilibria

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March 15, 2006

Toroidal flow velocities of the order of the local sound speed and poloidal flows exceeding theoretical predictions have been observed in several tokamaks. Steady toroidal and poloidal flow effects are studied using dissipationless single-fluid and two-fluid theory, with electron inertia neglected in the latter case. An exact analytic treatment of the two-fluid system, with the electron and ion temperatures both assumed to be flux functions and ion poloidal flows neglected, reveals a much wider class of rotation profiles than those corresponding to rigid body rotation of flux surfaces, which is required by ideal magnetohydrodynamics (MHD). A generalized expression is obtained for the variation of density on a flux surface in the presence of flows, and a relation is established between the rotation and temperature profiles that makes it possible to test experimentally the assumption of rigid body rotation. Relaxing the assumption that ion temperature is a flux function leads to a still wider class of possible profiles. It is shown that ion momentum balance in the absence of ion poloidal flows implies a Grad-Shafranov equation that is structurally similar to the standard ideal MHD form of this equation. Leading order ion poloidal flow corrections to the Grad-Shafranov equation are also computed.

I. Introduction

Tokamak plasma equilibria are traditionally modelled by equating the pressure gradient and Lorentz force terms in the magnetohydrodynamic (MHD) momentum equation, with inertial terms associated with toroidal or poloidal flows assumed to play no significant role. However, toroidal flow velocities comparable to or exceeding the local sound speed have recently been observed in the Mega-Ampère Spherical Tokamak (MAST) [1]; toroidal flows with sonic Mach numbers approaching unity have also been reported in DIII-D [2], the National Spherical Tokamak Experiment (NSTX) [3] and the Joint European Torus (JET) [4]. In this regime the inertial term in the MHD momentum equation is not negligible compared to the pressure gradient term, and one would therefore expect the equilibrium to be significantly modified. In the case of MAST, the discharges with the most rapid toroidal flows are heated by neutral beams injected counter to the plasma current direction. In such circumstances a high proportion of the beam ions are lost promptly, leading to the formation of large radial electric fields and hence rapid toroidal rotation [5]. Measurements of carbon impurity ions in the Tokamak Fusion Test Reactor (TFTR) [6] and, more recently, in DIII-D [2] and JET [7], have also revealed poloidal flows of up to several tens of kilometres per second, exceeding by a large factor the values predicted on the basis of neoclassical theory.

A number of authors have studied toroidal [8, 9] and poloidal [10, 11, 12, 13, 14] flow modifications to axisymmetric equilibria in the framework of ideal MHD. The present authors have investigated two-fluid axisymmetric equilibria with arbitrary electron and ion flows [15], deriving a set of equations that is analogous to the Grad-Shafranov-Bernoulli system of ideal MHD. The purpose of the present paper is to study in general terms the effects of steady toroidal and poloidal flows on tokamak plasma profiles and equilibrium magnetic field structure in the framework of both single-fluid (Sec. II) and two-fluid theory (Secs. III and IV), with the principal emphasis on the latter. It will be shown that the two-fluid viewpoint leads to new results (from which earlier results can be recovered in suitable limits) concerning the variation of electron density, pressure, electrostatic potential, and toroidal flow velocity. These results are testable experimentally, using data from tokamaks with strong flows, such as MAST, DIII-

D, NSTX and JET, and could thus be used in principle to deduce values of plasma parameters in those devices. Another motivation for studying the effects of steady flows in tokamaks is that it is important to have reasonably accurate equilibria corresponding to actual experimental conditions before examining them for stability: strong flows are known to influence instabilities in tokamaks, in some cases playing a key role in stabilising them and reducing transport losses (see e.g. [16]).

II. Single-fluid theory

Ideal MHD equilibria are represented by steady-state solutions of the single-fluid momentum balance equation

$$\rho_m \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B}, \tag{1}$$

and Ohm's law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0, \tag{2}$$

together with an equation of state relating pressure p and mass density ρ_m. In Eqs. (1) and (2) $\mathbf{v}$ is fluid velocity, d/dt denotes convective time derivative, $\mathbf{j}$ is current, $\mathbf{B}$ is magnetic field and $\mathbf{E}$ is electric field. A toroidally-symmetric magnetic field satisfying $\nabla \cdot \mathbf{B} = 0$ can be represented generally by the expression

$$\mathbf{B} = \left(-\frac{1}{R}\frac{\partial \Psi}{\partial Z}\right)\mathbf{e}_R + B_\phi \mathbf{e}_\phi + \left(\frac{1}{R}\frac{\partial \Psi}{\partial R}\right)\mathbf{e}_Z, \tag{3}$$

where $\mathbf{e}_R$, $\mathbf{e}_\phi$, $\mathbf{e}_Z$ are unit vectors in a right-handed (R, ϕ, Z) cylindrical coordinate system and Ψ, B_ϕ depend only on R and Z. We assume that all other dependent variables are also independent of ϕ. Under steady-state conditions and in the absence of plasma sources and sinks, the mass flux $\rho_m \mathbf{v}$ must also be divergence-free, and hence expressible in the form

$$\rho_m \mathbf{v} = \left(-\frac{1}{R}\frac{\partial \Theta}{\partial Z}\right)\mathbf{e}_R + \rho_m v_\phi \mathbf{e}_\phi + \left(\frac{1}{R}\frac{\partial \Theta}{\partial R}\right)\mathbf{e}_Z, \tag{4}$$

where v_ϕ is toroidal flow velocity and Θ is a stream function for the poloidal flow.

The steady-state assumption requires that $\nabla \times \mathbf{E} = \mathbf{0}$ and hence that there exist a function Φ such that $\mathbf{E} = -\nabla\Phi$. Under conditions of toroidal symmetry, the toroidal component of Eq. (2) then yields

$$\frac{\partial(\Theta, \Psi)}{\partial(R, Z)} = 0. \tag{5}$$

We infer that Θ depends only upon Ψ and write $\Theta = W(\Psi)$, where W is a function that describes the poloidal flow. Denoting the poloidal components of the flow and the magnetic field by v_θ and B_θ, it is clear from Eqs. (3) and (4) that

$$\frac{\rho_m v_\theta}{B_\theta} = W', \tag{6}$$

where the prime denotes differentiation with respect to Ψ. Thus, the ratio of poloidal mass flux to poloidal magnetic field must be a flux function. We note that once $\Psi(R, Z)$

and $\rho_m(R, Z)$ have been calculated and the flux function $W(\Psi)$ specified, both B_θ and v_θ are determined from the above formulae.

As noted in Ref. [15], it follows from these equations that the electrostatic potential Φ must be a flux function, irrespective of any ordering of the toroidal and poloidal flows, and that its derivative with respect to Ψ is given by

$$\Phi' = \frac{v_\phi}{R} - \left(\frac{B_\phi}{B_\theta}\right)\frac{v_\theta}{R}. \qquad (7)$$

In the case of conventional tokamaks, with $|B_\phi| \gg |B_\theta|$, we would thus expect poloidal flows to play a significant role in determining the radial electric field $E_\Psi \propto -\Phi'$ unless the order of v_θ/v_ϕ is even smaller than that of B_θ/B_ϕ. In general the flows are determined by a combination of transport processes and momentum sources (such as neutral particle beams) and sinks. In most instances it is unlikely that neoclassical processes will adequately describe the transport, although they will always play a role in determining the total momentum fluxes. It is of interest to observe that in MAST $B_\theta \simeq B_\phi$ and, with counter beam injection $v_\phi \simeq 300$ kms^{-1} [1], $v_\theta < 50$ kms^{-1} [17]. The above equation then implies that the toroidal plasma flow on each flux surface will be approximately that of a rigid body, although a variation of up to around 10% in the toroidal angular velocity can be expected due to the poloidal flow.

Assuming toroidal symmetry, one can deduce from the toroidal component of the momentum equation [Eq. (1)] the existence of a flux function f given by [15]

$$f(\Psi) \quad = \quad RB_\phi - \mu_0 W' R v_\phi, \tag{8}$$

where we have used Ampère's law, μ_0 being free space permeability. Equation (8) reduces to the familiar result $RB_\phi = f(\Psi)$ if either the poloidal or toroidal flow is zero. When such flows are present, we can solve for B_ϕ:

$$RB_\phi \quad = \quad \frac{f(\Psi) + \mu_0 R^2 \Phi' W'}{(1 - M_*^2)}, \tag{9}$$

where

$$M_* \quad = \quad \frac{v_\theta}{c_{A\theta}}, \tag{10}$$

is the poloidal flow normalised to the poloidal Alfvén speed

$$c_{A\theta} \quad = \quad \frac{B_\theta}{\sqrt{\mu_0 \rho_m}}. \tag{11}$$

We note from Eq. (6) that M_* varies on a flux surface if and only if ρ_m does. With the above definitions, we can rewrite Eq. (8) in the following perspicuous form:

$$RB_\phi \quad = \quad f(\Psi) + RB_\theta \left(\frac{v_\theta v_\phi}{c_{A\theta}^2} \right). \tag{12}$$

We can also obtain an expression for the toroidal angular velocity $\Omega_\phi \equiv v_\phi / R$:

$$\Omega_\phi \quad = \quad \frac{1}{1 - M_*^2} \left[\Phi' + \frac{f W'}{R^2 \rho_m} \right]. \tag{13}$$

This expression shows explicitly that in the limit of ideal MHD the toroidal angular velocity can vary on a flux surface (unlike the electrostatic potential) if poloidal flows are present. The flows v_ϕ and v_θ must satisfy momentum transport equations, the solution of which requires knowledge of the flux surfaces as well as the relevant transport coefficients and sources. In these circumstances, flux functions such as Φ, W and f can be obtained by solving relevant transport equations simultaneously with the equilibrium equations, given appropriate sources and boundary conditions. We have demonstrated that these three flux functions together with Ψ and ρ_m can be used to calculate the poloidal and toroidal magnetic fields, electric field and the flows. In particular, the variations of these fields on a given flux surface with R are completely determined.

An equation for ρ_m will now be obtained, assuming that the temperature T is a flux function. The pressure is then given by $p = \rho_m T(\Psi)/m$ where m is the mean particle mass and T is in energy units. In this case momentum balance in the direction parallel to $\mathbf{B}$ yields

$$\mathbf{B} \cdot \nabla \left[\frac{v_\phi^2}{2} + \frac{v_\theta^2}{2} - \Phi' R v_\phi + \frac{T(\Psi)}{m} \ln \rho_m \right] \quad = \quad 0. \tag{14}$$

Hence we deduce the existence of a flux function H such that the following Bernoulli relation applies:

$$\frac{v_\phi^2}{2} + \frac{v_\theta^2}{2} - \Phi' R v_\phi + \frac{T(\Psi)}{m} \ln \rho_m \quad = \quad H(\Psi). \tag{15}$$

Setting $v_\phi = v_\theta = 0$ in this equation we recover the well-known result that ρ_m is a flux function in the absence of flows if T is a flux function. Putting $v_\theta = W' B_\theta/\rho_m = W'|\nabla\Psi|/(R\rho_m)$ and $v_\phi = \Omega_\phi R$, with Ω_ϕ given by Eq. (13), the Bernoulli relation becomes a transcendental equation for ρ_m in terms of $\Psi(R, Z)$ and its derivatives, together with R and the flux functions W', Φ', f, T and H. Once this equation is solved, the variation of ρ_m with R and Z can be determined by solving a partial differential (Grad-Shafranov) equation for Ψ. It is important to note, however, that Eq. (15), which is valid for arbitrary v_ϕ and v_θ, is sufficient to determine the variation of ρ_m on a flux surface.

The transcendental equation for ρ_m becomes tractable when v_θ is negligible compared to v_ϕ. It follows from Eq. (15) that we can formally write

$$\rho_m \quad = \quad \bar{K}(\Psi) \exp\left[-\frac{m(v_\phi^2 + v_\theta^2 - 2\Phi' R v_\phi)}{2T(\Psi)}\right], \tag{16}$$

where $\bar{K} = \exp(mH/T)$ is a flux function. Taking the limit $v_\theta \to 0$ and substituting for Φ' from Eq. (13), we find that Eq. (16) reduces to

$$\rho_m \quad = \quad \bar{K}(\Psi) \exp\left[\frac{m\Omega_\phi^2(\Psi) R^2}{2T(\Psi)}\right], \tag{17}$$

where the toroidal rotation rate $\Omega_\phi = \Phi'$ is a flux function in this limit. Under conditions of quasi-neutrality, Eq. (17) gives the variation of electron density on a flux surface. This result for the density contrast was obtained by Wesson [18]. If single-fluid theory is assumed to be applicable, with T isotropic and a flux function, any measured departure of the density variation from that indicated by Eq. (17) must be attributed to poloidal flows. When the flux surfaces are assumed to be isentropic rather than isothermal, the predicted flux surface variation of density in the presence of purely toroidal flows differs from that given by Eq. (17), and there is also a variation of temperature [8]. We will show later that two-fluid analysis implies variations of quantities on flux surfaces that differ, in general, from the predictions of MHD. Thus it is possible, in principle, to distinguish between the two models experimentally.

By examining the components of the momentum equation in the (R, Z) plane, eliminating $\mathbf{j}$ via Ampère's law and making use of the flux functions deduced above, it is straightforward to derive a generalised single-fluid Grad-Shafranov equation for arbitrary flows, with either T or entropy assumed to be a flux function [10, 11, 15]. When T is a flux function the equation can be written in the form

$$\begin{aligned} \frac{\partial}{\partial Z}(\Delta\frac{\partial\Psi}{\partial Z}) + R\frac{\partial}{\partial R}(\Delta\frac{1}{R}\frac{\partial\Psi}{\partial R}) &= -\mu_0 R^2(\frac{\rho_m}{\bar{K}})P' - RB_\phi f' + \mu_0 R^2\rho_m(\frac{T'}{m})\ln(\frac{\rho_m}{\bar{K}}) \\ &\quad -\mu_0 W'W''\frac{|\nabla\Psi|^2}{\rho_m} \\ &\quad -\mu_0\rho_m R^2\Phi'' Rv_\phi - \mu_0(RB_\phi)(Rv_\phi)W'', \end{aligned} \tag{18}$$

where $P(\Psi) \equiv \bar{K}(\Psi)T(\Psi)/m$ and

$$\begin{aligned} \Delta &= 1 - \frac{\mu_0(W')^2}{\rho_m} \\ &= 1 - M_*^2. \end{aligned} \tag{19}$$

When $v_\phi = v_\theta = 0$ we have $\Delta = 1$, $\rho_m = \bar{K}(\Psi)$, $p = P(\Psi)$ and $RB_\phi = f(\Psi)$: it is evident that Eq. (18) then reduces to the familiar form of the Grad-Shafranov equation for tokamak equilibria without flows [19]. In general, the equation must be solved numerically with specified boundary data and external coil currents. This is a formidable problem, even when singularities due to $\Delta \to 0$ do not arise and the equation remains elliptic everywhere. Such a solution is needed for complete consistency in the determination of ρ_m, the flow velocities and $\Psi(R, Z)$. Equation (18) does, however, simplify considerably in the limit $v_\theta \to 0$, to such an extent that nontrivial analytical solutions can be constructed [8, 9]. Such solutions may provide an approximate description of MAST plasmas with neutral beam injection in the direction counter to that of the plasma current [1].

III. Two-fluid theory: purely toroidal ion flows

In this section we discuss quasi-neutral, toroidally-symmetric solutions of the steady-state dissipationless two-fluid equations

$$\nabla \cdot (n\mathbf{v}_{i,e}) = 0, \tag{20}$$

$$m_{i,e}\mathbf{K}_{i,e} \times n\mathbf{v}_{i,e} = -\nabla (nT_{i,e}) - m_{i,e}n\nabla \left(\mathbf{v}_{i,e}^2/2\right) - e_{i,e}n\nabla\Phi + e_{i,e}n\mathbf{v}_{i,e} \times \mathbf{B}, \tag{21}$$

$$\nabla \times \mathbf{B} = \mu_0 en(\mathbf{v}_i - \mathbf{v}_e). \tag{22}$$

Here n is the common number density of ions and electrons, labelled respectively by the suffixes i and e on other quantities; m, e, $\mathbf{v}$ and T denote respectively particle mass, particle charge, fluid velocity and temperature (assumed to be isotropic for both species); and $\mathbf{K} \equiv \nabla \times \mathbf{v}$ is vorticity. Closure of Eqs. (20-22) is provided by energy equations for the two species plus $\nabla \cdot \mathbf{B} = 0$.

The assumption of isotropic temperature is usually justified for bulk electrons and ions in tokamaks. Moreover, parallel electron heat transport is sufficiently rapid that T_e can be treated as a flux function to a high degree of accuracy; we shall do so, thereby obviating the need for an electron energy balance equation. On the other hand it is by no means clear that T_i should be a flux function, especially when there are large flows present. In a dissipationless framework one might reasonably assume that n and T_i are isentropically related. We discuss the ion energy equation in an Appendix, leaving its variation on a flux surface to be arbitrary for the present. The important special case of T_i being constant on a flux surface will be discussed later in some detail.

We now proceed to reduce Eqs. (20-22), making appropriate use of the ordering $m_e/m_i \rightarrow 0$. A more general reduction was carried out by the present authors in [15]. However, the equations derived in that paper are not analytically tractable and are more complicated to solve than the single-fluid model presented in the preceding section.

Due to toroidal symmetry the magnetic field can be represented by Eq. (3), as in the case of ideal MHD. Two-fluid mass conservation [Eq. (20)] implies moreover the existence of poloidal flow stream functions $\Theta_{i,e}(R,Z)$ such that

$$\begin{aligned} n\mathbf{v}_{i,e} &= (-\frac{1}{R}\frac{\partial\Theta_{i,e}}{\partial Z})\mathbf{e}_R + nv_\phi^{i,e}\mathbf{e}_\phi + (\frac{1}{R}\frac{\partial\Theta_{i,e}}{\partial R})\mathbf{e}_Z, \\ &= \nabla\Theta_{i,e}\times\nabla\phi + nRv_\phi^{i,e}\nabla\phi, \end{aligned} \tag{23}$$

where $v_\phi^{i,e}(R,Z)$ are the toroidal flow velocities. As in the case of ideal MHD, we will show that certain flux functions can be used to determine all other variables (n, $\mathbf{v}_{i,e}$, T_i, Φ, $\mathbf{B}$) in terms of R, together with Ψ and its first derivatives. The poloidal flux function itself will be shown to satisfy a generalized Grad-Shafranov equation, consistent with the two-fluid system of equations. In principle, the flux functions could be determined by solving transport equations with specified sources, boundary data and suitable turbulence-driven and/or collisional transport coefficients. Alternatively, experimental data could be used to infer their form.

We begin our analysis with the electrons, as in this case the equations simplify considerably in the limit $m_e/m_i \to 0$. Neglecting inertia and dissipation, the electron momentum balance equation takes the simple form

$$0 = -\nabla nT_e + en\nabla\Phi - en\mathbf{v}_e\times\mathbf{B}, \tag{24}$$

where e is the proton charge. With $\mathbf{B}$ and $n\mathbf{v}_e$ represented by the expressions in Eqs. (3) and (23), one can deduce from the toroidal component of the full electron momentum balance equation (with inertial terms retained) that the toroidal canonical momentum of the electron fluid $P_\phi^e \equiv m_e R v_\phi^e - e\Psi$ is a function of Θ_e under conditions of toroidal symmetry [15]. It follows that Θ_e must be a flux function in the limit of negligible electron inertia. This result is generally an excellent approximation in MAST: the electron mechanical toroidal momentum is typically around three orders of magnitude smaller than $e\Psi$. Thus, the electron fluid can only move within flux surfaces, and significant cross-field fluxes cannot occur in the absence of large toroidal symmetry-breaking force.

With T_e taken to be a flux function, the component of Eq. (24) parallel with $\mathbf{B}$ yields the adiabaticity relation

$$T_e \ln n \quad = \quad e\Phi + h_e(\Psi), \tag{25}$$

where $h_e(\Psi)$ is a flux function. Thus, there can be variations of n on a flux surface if and only if there are such variations in the electrostatic potential, Φ. Direct measurements of the latter, although difficult, would be of great value. Eliminating Φ from Eq. (24) using Eq. (25) we obtain the following equation describing electron force balance in the direction normal to the flux surface:

$$T_e' \ln n \quad = \quad (T_e + h_e)' + \frac{e v_\phi^e}{R} - \left[\frac{e B_\phi \Theta_e'}{nR}\right]. \tag{26}$$

We will make use of this relation later.

Before considering ion dynamics, we obtain two useful relations from Ampère's law [Eq.(22)]. The poloidal component of this equation can be integrated exactly to give

$$RB_\phi \;=\; R_0 B_{\phi 0} + e\mu_0(\Theta_i - \Theta_e). \tag{27}$$

where $R_0 B_{\phi 0}$ is an integration constant that may be taken to be RB_ϕ at the magnetic axis, $R = R_0$, $Z = 0$. We have seen that Θ_e is a flux function; in general Θ_i is not a flux function [15]. Equation (27) enables RB_ϕ to be eliminated in favour of these quantities, describing ion and electron poloidal flows. Using Eq. (3), we can write the toroidal component of Eq.(22) in the form

$$j_\phi \;=\; -\frac{1}{\mu_0 R}\left[\frac{\partial^2\Psi}{\partial Z^2} + R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R}\right)\right]. \tag{28}$$

where $j_\phi \equiv en(v_\phi^i - v_\phi^e)$ is the toroidal current density.

We now consider ion momentum balance. If no assumptions are made regarding the relative sizes of toroidal and poloidal ion flows, one can derive a rather complicated second-order nonlinear elliptic partial differential equation for Θ_i and a Bernoulli equation referred to surfaces of constant Θ_i (rather than constant Ψ) [15]. In addition, toroidal symmetry requires that $P_\phi^i \equiv mRv_\phi^i + e\Psi$ be a function only of Θ_i. These predictions are essentially different from (and more complicated than) the single-fluid, ideal MHD results of the preceding section. However, both theoretical and experimental studies tend to suggest that while the ion poloidal flows may not always be as small as those suggested by neoclassical considerations (cf. [20], p. 366), they are nevertheless somewhat smaller than toroidal flows, which, in the case of MAST discharges with counter current neutral beam injection, can exceed the ion acoustic speed. We therefore concentrate in this paper on the experimentally-relevant limit $|v_\theta^i| \ll |v_\phi^i|$. We first neglect poloidal ion flows entirely, i.e. we set $\Theta_i = 0$. Later we will indicate how poloidal ion flows can be treated using a perturbation expansion in the small parameter $|v_\theta^i/v_\phi^i|$.

With $\Theta_i = 0$ it follows immediately from Eq. (27) that $RB_\phi \equiv f(\Psi)$ is a flux function. Summing the electron and ion momentum equations we obtain

$$m_i n \mathbf{K}_i \times \mathbf{v}_i \;\; = \;\; -\nabla p_{\mathrm{tot}} - m_i n \nabla(\mathbf{v}_i^2/2) + \mathbf{j} \times \mathbf{B}, \tag{29}$$

where $p_{\mathrm{tot}} = 2nT$, $T = (T_i + T_e)/2$. For the case of a purely toroidal flow $\mathbf{v}_i = Rv_\phi^i \nabla\phi$ the vorticity is

$$\mathbf{K}_i \;\; = \;\; \nabla R v_\phi^i \times \nabla\phi. \tag{30}$$

Using Eq. (30) together with Ampère's law in Eq. (29) we obtain

$$\nabla p_{\mathrm{tot}} \;\; = \;\; \frac{m_i n (v_\phi^i)^2}{R} \nabla R + \left[\frac{R\mu_0 j_\phi - f f'}{\mu_0 R^2}\right] \nabla\Psi. \tag{31}$$

This equation indicates that in steady state the plasma pressure gradient balances the sum of a centrifugal force, acting along the major radial direction, and a Lorentz force acting normal to the flux surface. The pressure cannot therefore be a flux function, as it is in the absence of flows. The poloidal component of Eq. (31) can be written in the form

$$\frac{\partial p_{\rm tot}}{\partial l} = \frac{m_i n (v_\phi^i)^2}{R}\frac{\partial R}{\partial l}, \tag{32}$$

where l denotes arc length along a flux surface in the (R, Z) plane. Dividing both sides by $p_{\rm tot} = 2nT$ and using the definition $v_\phi^i = \Omega_\phi^i R$, the equation can be recast in the form

$$\frac{\partial \ln p_{\rm tot}}{\partial l} = \frac{m_i (\Omega_\phi^i)^2}{4T}\frac{\partial R^2}{\partial l}. \tag{33}$$

This equation can be satisfied in an infinite number of ways, but we consider only physically transparent cases. Let us regard Ψ and R^2 as independent variables describing a poloidal plane. This is permissible, since the Jacobian of Ψ and R^2 does not generally vanish. Introducing a function $V(\Psi, R^2)$ such that

$$\frac{m_i (\Omega_\phi^i)^2}{4T} = \frac{\partial V}{\partial R^2}, \tag{34}$$

Eq. (33) can be formally integrated to yield the Bernoulli relation

$$\ln p_{\rm tot} - V(\Psi, R^2) = \ln P^*(\Psi), \tag{35}$$

where $P^*(\Psi)$ is a flux function. Hence

$$p_{\rm tot} = P^*(\Psi) \exp\left[V(\Psi, R^2)\right]. \tag{36}$$

Since P^* is a function of Ψ only, we can rewrite Eq. (34) in the form

$$\frac{m_i(\Omega^i_\phi)^2}{4T} = \frac{\partial \ln p_{\rm tot}}{\partial R^2}. \tag{37}$$

Writing

$$\nabla p_{\rm tot} = \frac{\partial p_{\rm tot}}{\partial R^2}\nabla R^2 + \frac{\partial p_{\rm tot}}{\partial \Psi}\nabla \Psi, \tag{38}$$

and comparing with Eq. (31), we deduce that

$$\frac{\partial p_{\rm tot}}{\partial \Psi} = \frac{j_\phi}{R} - \frac{ff'}{\mu_0 R^2}. \tag{39}$$

Now Eq. (26) can be rewritten as follows, using $RB_\phi = f = R_0 B_{\phi 0} - e\mu_0\Theta_e$:

$$T'_e n \ln n = n(T_e + h_e)' + \frac{env^e_\phi}{R} + \left[\frac{ff'}{\mu_0 R^2}\right]. \tag{40}$$

Adding Eqs. (39) and (40), and using the definitions of j_ϕ and v^i_ϕ, we obtain

$$T'_e n \ln n - n(T_e + h_e)' + \frac{\partial p_{\rm tot}}{\partial \Psi} = en\Omega^i_\phi. \tag{41}$$

Dividing by $p_{\rm tot} = 2nT$, we then obtain

$$\frac{T'_e}{2T}\ln p_{\rm tot} - \frac{T'_e}{2T}\ln 2T - \frac{(T_e + h_e)'}{2T} + \frac{\partial \ln p_{\rm tot}}{\partial \Psi} = \frac{e\Omega^i_\phi}{2T}. \tag{42}$$

We now show that the system can be closed by specifying T_i through an equation of state. As an example, we consider the important special case when T_i is assumed to be a flux function (cf. Appendix). Then, T is also a flux function. Differentiating Eq.

(42) with respect to R^2 and eliminating $p_{\rm tot}$ using Eq. (37) we obtain

$$\frac{T_e'}{T}\frac{m_i(\Omega_\phi^i)^2}{8T} + \frac{\partial}{\partial\Psi}\left(\frac{m_i(\Omega_\phi^i)^2}{4T}\right) = \frac{e}{2T}\frac{\partial\Omega_\phi^i}{\partial R^2}. \tag{43}$$

Before computing the complete solution of this equation, we consider two interesting special cases. Let us suppose that the toroidal angular velocity of the ions is a flux function: the ion fluid on a flux surface then rotates toroidally as a rigid body. Equation (43) then reduces to a linear ordinary differential equation

$$\frac{T_e'}{T}\frac{m_i(\Omega_\phi^i)^2}{8T} + \frac{d}{d\Psi}\left(\frac{m_i(\Omega_\phi^i)^2}{4T}\right) = 0, \tag{44}$$

with solution

$$\frac{m_i(\Omega_\phi^i)^2}{4T} = \frac{m_i(\Omega_{\phi 0}^i)^2}{4T_0}\exp\left[-\int_{\Psi_0}^{\Psi}\frac{T_e' d\psi}{2T}\right], \tag{45}$$

where Ψ_0 is the value of Ψ at some reference surface (for example, the magnetic axis) and the subscript 0 denotes values at this location. Since $V = m_i(\Omega_\phi^i R)^2/(4T)$ in this case [cf. Eq. (34)], it follows that

$$V(\Psi, R^2) = \frac{m_i(\Omega_{\phi 0}^i)^2 R^2}{4T_0}\exp\left[-\int_{\Psi_0}^{\Psi}\frac{T_e' d\psi}{2T}\right]. \tag{46}$$

Moreover from Eq. (36) we obtain

$$p_{\rm tot} = P^*(\Psi)\exp\left(\frac{m_i(\Omega_\phi^i R)^2}{4T}\right), \tag{47}$$

and

$$n = N^*(\Psi)\exp\left(\frac{m_i(\Omega_\phi^i R)^2}{4T}\right), \tag{48}$$

where $2N^*(\Psi)T(\Psi) \equiv P^*(\Psi)$. We deduce finally from the electron adiabaticity relation [Eq. (25)] that

$$\frac{e\Phi}{T_e} = \frac{e\Phi^*(\Psi)}{T_e} + \frac{m_i(\Omega_\phi^i R)^2}{4T}, \tag{49}$$

where $e\Phi^* = T_e \ln N^* - h_e$.

It should be noted that although these results resemble the ideal MHD results of the previous section insofar as $p_{\rm tot}$ and n have a Gaussian dependence on R, two-fluid theory completely determines the angular velocity Ω_ϕ^i in terms of the specified electron and ion temperature variations with respect to Ψ [Eq. (45)]. In ideal MHD the angular velocity can be an arbitrary function of Ψ, whatever the specified variation of T. Moreover, two-fluid theory implies an electric potential variation on a flux surface due to purely toroidal flow [Eq. (49)], whereas in ideal MHD Φ remains a flux function in the presence of arbitrary toroidal or poloidal flows. In principle, Eq. (45) can be tested experimentally by measuring $T(\Psi)$ and $\Omega_\phi^i(\Psi)$. Such measurements could also test the hypotheses that T and Ω_ϕ^i are indeed flux functions for arbitrary toroidal ion flows (including transonic flows).

A second special case is that in which the mechanical toroidal angular momentum per unit mass of the ion fluid is a flux function, i.e. the ions rotate toroidally in a Keplerian manner. We can then write $\Omega_\phi^i = \lambda(\Psi)/R^2$: clearly the toroidal canonical momentum of the ion fluid $P_\phi^i = m_i\Omega_\phi^i R^2 + e\Psi$ is also a flux function in this case. From Eq. (43)

we find that λ satisfies the ordinary differential equation

$$\frac{T_e'}{2T}\frac{m_i\lambda^2}{4T} + \frac{d}{d\Psi}\left(\frac{m_i\lambda^2}{4T}\right) \quad = \quad -\frac{e\lambda}{2T}.$$

This reduces to a linear equation

$$\frac{d\lambda}{d\Psi} - \frac{T_i'}{4T}\lambda \quad = \quad -\frac{e}{m_i}. \tag{50}$$

with solution

$$\lambda(\Psi) \quad = \quad \frac{1}{\mu(\Psi)}\left[\lambda_0 - \frac{e}{m_i}\int_{\Psi_0}^{\Psi}\mu(\psi)d\psi\right], \tag{51}$$

where

$$\mu(\Psi) \quad = \quad \exp\left[-\int_{\Psi_0}^{\Psi}\frac{T_i'd\psi}{4T}\right],$$

and $\lambda_0 = \lambda(\Psi_0)$. For specified ion and electron temperature profiles Eq. (51) gives the variation of ion toroidal angular momentum with Ψ. We can also obtain expressions for V, $p_{\rm tot}$, n and Φ in terms of λ:

$$V(\Psi, R^2) \quad = \quad -\frac{m_i\lambda^2}{4TR^2}, \tag{52}$$

$$p_{\rm tot} \quad = \quad P^*(\Psi)\exp(-\frac{m_i\lambda^2}{4TR^2}), \tag{53}$$

$$n \quad = \quad N^*(\Psi)\exp(-\frac{m_i\lambda^2}{4TR^2}), \tag{54}$$

$$\frac{e\Phi}{T_e} \quad = \quad \frac{e\Phi^*(\Psi)}{T_e} - \frac{m_i\lambda^2}{4TR^2}. \tag{55}$$

This model predicts a pressure and density contrast on a flux surface, but differs from the previous rigid body solution [cf. Eqs. (46)-(49)] in important respects. Both solutions predict that the density and pressure will be larger on the outboard side of the plasma, although the actual variation with R^2 is different in the two cases. More importantly, in the rigid body solution there is, by definition, no variation of the angular velocity on a flux surface, whereas in the Keplerian solution the toroidal angular angular and the toroidal velocity are higher on the inboard side of a given flux surface.

We now compute the complete solution of Eq. (43), assuming only that T is a flux function. Defining a new dependent variable $\zeta^2 = m_i(\Omega^i_\phi)^2/4T$, a new independent variable $y = R^2$, and a function $\xi(\Psi)$ by the expression

$$\xi(\Psi) = \exp\left[\int_{\Psi_0}^{\Psi} \frac{T_e' d\psi}{2T}\right] \tag{56}$$

$$= \frac{T}{T_0}\mu^2, \tag{57}$$

we find that Eq. (43) can be written in the form

$$\frac{\partial}{\partial\Psi}\left[\xi\zeta^2\right] = \frac{\partial}{\partial y}\left[\frac{e\xi\zeta}{\sqrt{m_i T}}\right].$$

Setting $z = \xi^{1/2}\zeta$ and

$$x = \int_{\Psi_0}^{\Psi} \frac{e}{\sqrt{m_i T}}\xi(\psi)d\psi, \tag{58}$$

it becomes apparent that the equation for ζ reduces to the simple form

$$\frac{\partial z^2}{\partial x} = \frac{\partial z}{\partial y}. \tag{59}$$

This first order nonlinear partial differential equation can be integrated using standard techniques [21]. The complete integral is

$$z = \frac{c - x}{2(y + d)}, \tag{60}$$

where c and d are arbitrary constants. An infinity of solutions can be obtained from this two-parameter family of complete integrals. We obtain nontrivial solutions that are independent of y by putting $c = kd$, where k is an arbitrary constant, and taking the limit $d \to \infty$. These solutions, which correspond to the rigid body rotation case considered above, are also independent of x: setting $k = (m_i/T_0)^{1/2}\Omega^i_{\phi 0}$, we recover Eq. (45). Setting $c = m_i^{1/2}\lambda_0/T_0^{1/2}$, $d = 0$, on the other hand, we recover the Keplerian solution [Eq. (51)].

In the general case the Ψ dependence of the solution is complicated and depends on the temperature profiles, but the R^2 dependence of $p_{\rm tot}$ and n is uniquely determined by the complete integral given by Eq. (60). Equation (37), when written in the form

$$\frac{\partial \ln p_{\rm tot}}{\partial y} = \frac{A(\Psi)^2}{(R^2 + d)^2}, \tag{61}$$

where

$$A(\Psi) = \frac{c - x(\Psi)}{2\xi^{1/2}(\Psi)}, \tag{62}$$

can be easily integrated to give

$$p_{\rm tot} = P^*(\Psi) \exp\left[-\frac{A(\Psi)^2}{(R^2 + d)}\right]. \tag{63}$$

Hence we deduce that

$$n = N^*(\Psi)\exp\left[-\frac{A(\Psi)^2}{(R^2+d)}\right], \tag{64}$$

and, from Eq. (25),

$$\frac{e\Phi}{T_e} = \frac{e\Phi^*(\Psi)}{T_e} - \frac{A(\Psi)^2}{(R^2+d)}. \tag{65}$$

The rotation profile in the general case is given by

$$\Omega_\phi^i = \left(\frac{4T(\Psi)}{m_i}\right)^{1/2}\frac{A(\Psi)}{R^2+d}. \tag{66}$$

It should be noted that Eqs. (63) - (66) are exact results obtained from the steady two-fluid system of equations in the limit $m_e/m_i \to 0$, with T_e and T_i taken to be flux functions and v_θ^i taken to be negligible. They are clearly more general than the ideal MHD results derived earlier, and include them as a special case: expanding $-1/(R^2+d)$ for $d \to \infty$ and putting $m = (m_i + m_e)/2 \simeq m_i/2$, we recover from Eqs. (64) and (66) the density variation given by Eq. (17).

If we relax the assumption that T_i is a flux function the theory can be used in principle to predict the ion temperature contrast on a flux surface. Equations (37) and (42), together with a specified energy equation for T_i, determine Ω_ϕ^i, $p_{\rm tot}$ and n as functions of R^2 and Ψ in terms of the specified flux functions. In general the ion equation of state is complicated and involves Φ, which in two-fluid theory is not a flux function. In view of this, we adopt a more experimentally-oriented approach to the case in which T_i

is not a flux function. If $p_{\rm tot}$ is specified as a function of Ψ and R^2, in accordance with experimental measurements, T_i and Ω_ϕ^i can then be computed using Eqs. (37) and (42). In such a procedure T_i is effectively determined by momentum balance rather than the energy equation, which is thus not required. We illustrate the method by expanding $\ln p_{\rm tot}$, Ω_ϕ^i and T as follows:

$$\ln p_{\rm tot} - \ln P^*(\Psi) = \left[\sigma_1(\Psi)y + \sigma_{-1}(\Psi)y^{-1} + \sigma_{-2}(\Psi)y^{-2} + ..\right], \tag{67}$$

$$\frac{1}{T} = \frac{1}{T_0(\Psi)}\left[1 + \theta_{-1}(\Psi)y^{-1} + \theta_{-2}(\Psi)y^{-2} + ..\right], \tag{68}$$

$$\Omega_\phi^i = \Omega_0^i(\Psi)\left[1 + \omega_{-1}(\Psi)y^{-1} + \omega_{-2}(\Psi)y^{-2} + ..\right], \tag{69}$$

where the coefficient functions σ_j, θ_j and ω_j are to be determined (the expansion parameter $y = R^2$ can be made dimensionless by normalising R to R_0, for example). The first terms on the right hand sides of Eqs. (67) - (69) correspond to the rigid body solution discussed previously, with T_i a flux function. Inserting these series expansions into Eq. (37) and equating like powers of y, we obtain the recurrence relations

$$\sigma_1 = \frac{m(\Omega_0^i)^2}{4T_0} \tag{70}$$

$$0 = \sigma_1(2\omega_{-1} + \theta_{-1}) \tag{71}$$

$$\sigma_{-1} = -\sigma_1(2\omega_{-2} + \omega_{-1}^2 + \theta_{-2} + 2\omega_{-1}\theta_{-1}) \tag{72}$$

$$\ldots \quad ..$$

Substituting the expansions into Eq. (42) we obtain

$$
\begin{aligned}
\frac{T_e'}{2T_0}\sigma_1 + \sigma_1' &= 0 && (73)\\
\frac{T_e'\sigma_1\theta_{-1}}{2T_0} &= \frac{e\Omega_0^i}{2T_0} + \frac{(T_e+h_e)'}{2T_0} - (\ln P^*)' - \frac{T_e'}{2T_0}\ln(\frac{P^*}{2T_0}). && (74)\\
\ldots &\quad .. &&
\end{aligned}
$$

The σ_j can be obtained directly from the function $p_{\text{tot}}(\Psi, R^2)$, since this is assumed to be determined experimentally. Equation (74) can then be used to obtain θ_{-1} if P^* and h_e are specified, and Eq. (71) then yields ω_{-1}. In a similar fashion θ_{-2}, ω_{-2} and so on can be determined self-consistently using the above recurrence relations. Alternatively, the coefficients could in priinciple be determined for any specified ion equation of state.

Finally in this section, we discuss briefly the two-fluid Grad-Shafranov equation in the absence of poloidal ion flows. This can be obtained very simply by rearranging Eq. (39), using the expression for j_ϕ given by Eq. (28):

$$\frac{\partial^2\Psi}{\partial Z^2} + R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R}\right) = -ff' - \mu_0 R^2\frac{\partial p_{\text{tot}}}{\partial\Psi}. \qquad (75)$$

Evidently this reduces to the ideal MHD Grad-Shafranov equation in the absence of flows when $p_{\text{tot}} = P^*(\Psi)$ [19]. In the general case, p_{tot} must be specified as a function of Ψ and R^2; if T_i and T_e are taken to be flux functions, p_{tot} is given by Eq. (63). With this important modification, the Grad-Shafranov equation, together with appropriate boundary data, can be used to determine $\Psi(R, Z)$ in the usual manner.

IV. Two-fluid equilibria: finite poloidal ion flows

In this section we briefly discuss a simple perturbative extension of the preceding theory when v_θ^i is assumed to be finite but small compared to v_ϕ^i: we do not assume, however, that the poloidal flows are necessarily as small as those predicted by neoclassical theory. Experimental data from several tokamaks suggest that it is appropriate to consider this scenario [2, 6, 7]. The ion vorticity $\mathbf{K}_i$ can be written as

$$\mathbf{K}_i \quad = \quad \nabla(Rv_\phi^i) \times \nabla\phi + RK_\phi^i \nabla\phi, \tag{76}$$

where

$$K_\phi^i \quad = \quad -\frac{1}{R}\left[\frac{\partial}{\partial Z}\left(\frac{1}{n}\frac{\partial \Theta_i}{\partial Z}\right) + R\frac{\partial}{\partial R}\left(\frac{1}{nR}\frac{\partial \Theta_i}{\partial R}\right)\right]. \tag{77}$$

Setting $RB_\phi = f$ as before, but without assuming that this is necessarily a flux function when poloidal ion flows are present, we find that the ion momentum balance equation takes the form

$$-\frac{m_i n (v_\phi^i)^2}{R}\nabla R + \frac{f}{\mu_0 R^2}\nabla f + \nabla p_{\mathrm{tot}} - \frac{j_\phi}{R}\nabla\Psi \quad = \quad -\frac{m_i n}{2}\nabla\left(\frac{\nabla\Theta_i}{n}\right)^2 - \frac{m_i K_\phi^i}{R}\nabla\Theta_i. \tag{78}$$

We now consider transonic flows, such that $(v_\phi^i)^2 \simeq T/m_i$, that satisfy the ordering $\Omega_\phi^i \ll eB_\theta/m_i \equiv \Omega_\theta^{ci}$ and $v_\theta^i \ll v_\phi^i$ where v_θ^i is the ion poloidal flow. In this regime the terms on the left hand side of Eq. (78) are formally all of the same order. The terms on the right however, are of order $(v^i_\theta/v^i_\phi)^2$ smaller. In general, as we have seen earlier, Ω^i_ϕ can be expressed as a function of Ψ and R^2. Using the definition $P^i_\phi = m_i\Omega^i_\phi R^2 + e\Psi$ we can write

$$\begin{aligned} P^i_\phi &= m_i\Omega^i_\phi R^2 + e\Psi \\ &= \langle P^i_\phi\rangle + \delta P, \end{aligned} \tag{79}$$

where angled brackets denote a flux surface average and $\langle P^i_\phi\rangle = m_i\langle\Omega^i_\phi R^2\rangle + e\Psi$, $\delta P = m_i(\Omega^i_\phi R^2 - \langle\Omega^i_\phi R^2\rangle)$. By construction, δP cannot be a flux function (except in the trivial case $\delta P = 0$), but it is of order $|\Omega^i_\phi|/\Omega^{ci}_\theta$ smaller than the flux function quantity $\langle P^i_\phi\rangle$. Note that $\delta P \to 0$ at the magnetic axis, whereas $\langle P^i_\phi\rangle$ is finite there (even if Ψ is defined such that it vanishes at that point). Since toroidal symmetry requires P^i_ϕ to be a function of Θ_i [15], there exists a function W^i such that $\Theta_i = W^i(P^i_\phi)$. Expanding to leading order in δP, we obtain

$$\begin{aligned} \Theta_i &= W^i(\langle P^i_\phi\rangle + \delta P) \\ &= W^i_0(\Psi) + W^i_1(\Psi)\delta P \end{aligned} \tag{80}$$

where $W^i_0(\Psi) = W^i(\langle P^i_\phi\rangle)$ and $W^i_1(\Psi) = dW^i/dP^i_\phi$, evaluated for $P^i_\phi = \langle P^i_\phi\rangle$. We also have the assumed small poloidal flow ordering $W^i_0 \simeq \epsilon\langle n\Omega^i_\phi R\rangle$ where $\epsilon = |v^i_\theta/v^i_\phi|$. This shows that, to leading orders in the assumed small parameters, Θ_i and hence f are flux functions. Note that the analysis is applicable for arbitrary plasma beta, ion toroidal

Mach number and aspect ratio. With Θ_i assumed to be a flux function, Eq. (78) can be written as

$$-\frac{m_i n(\Omega^i_\phi)^2}{2}\nabla R^2 + \frac{ff'}{\mu_0 R^2}\nabla\Psi + \nabla p_{\rm tot} - \frac{j_\phi}{R}\nabla\Psi \;=\; -\frac{m_i n}{2}\nabla\left(\frac{\Theta_i'\nabla\Psi}{n}\right)^2 - \frac{m_i K^i_\phi \Theta_i'}{R}\nabla\Psi. \tag{81}$$

We now introduce the flux surface average

$$\begin{aligned} v_\theta^2(\Psi) &\equiv \langle\left(\frac{\nabla\Theta_i}{n}\right)^2\rangle \\ &\simeq \langle\left(\frac{\Theta_i'\nabla\Psi}{n}\right)^2\rangle. \end{aligned} \tag{82}$$

Since the first term on the right hand side of Eq. (81) is assumed to be small, $(\Theta_i'\nabla\Psi/n)^2$ can be replaced by its flux surface average:

$$-\frac{m_i n(\Omega^i_\phi)^2}{2}\nabla R^2 + \frac{ff'}{\mu_0 R^2}\nabla\Psi + \nabla p_{\rm tot} - \frac{j_\phi}{R}\nabla\Psi \;=\; -\left[m_i n v_\theta v_\theta' + \frac{m_i K^i_\phi \Theta_i'}{R}\right]\nabla\Psi. \tag{83}$$

Writing $\nabla p_{\rm tot}$ in the form given by Eq. (38), and equating coefficients of ∇R^2 in Eq. (83), we obtain Eq. (37), as before. Equating coefficients of $\nabla\Psi$, on the other hand, we obtain a modified form of Eq. (39):

$$\frac{\partial p_{\rm tot}}{\partial\Psi} \;=\; \frac{j_\phi}{R} - \frac{ff'}{\mu_0 R^2} - p_{\rm tot}\left[\frac{m_i v_\theta v_\theta'}{2T} + \frac{m_i K^i_\phi \Theta_i'}{R p_{\rm tot}}\right]. \tag{84}$$

Adding this to Eq. (40) and dividing the sum by $p_{\rm tot}$ we obtain

$$\frac{T_e'}{2T}\ln p_{\rm tot} - \frac{T_e'}{2T}\ln 2T - \frac{(T_e+h_e)'}{2T} + \frac{\partial\ln p_{\rm tot}}{\partial\Psi} \;=\; \frac{e\Omega^i_\phi}{2T} - \left[\frac{m_i v_\theta v_\theta'}{2T} + \frac{m_i K^i_\phi \Theta_i'}{R p_{\rm tot}}\right] \tag{85}$$

It is apparent from Eq. (84) that the Grad-Shafranov equation now takes the form

$$\frac{\partial^2\Psi}{\partial z^2} + R\frac{\partial}{\partial R}(\frac{1}{R}\frac{\partial\Psi}{\partial R}) = -ff' - \mu_0 R^2\frac{\partial p_{\rm tot}}{\partial\Psi} - \mu_0 R^2 p_{\rm tot}\left[\frac{m_i v_\theta v_\theta'}{2T} + \frac{m_i K_\phi^i \Theta_i'}{R p_{\rm tot}}\right]. \quad (86)$$

This reduces, as required, to Eq. (75) in the limit $\Theta_i \to 0$. The right hand side of Eq. (86) includes a toroidal centrifugal term (represented by the R^2 dependence of $p_{\rm tot}$) and smaller poloidal flow terms. It should be noted that the flux function $f = RB_\phi$ is related to both the ion and electron poloidal flows via the poloidal component of Ampère's law [cf. Eq. (27)]. The above system of equations is valid for arbitrary toroidal and poloidal flows satisfying the ordering $|v_\theta^i/v_\phi^i| \to 0$.

V. Conclusions and discussion

We have considered toroidal and poloidal flow effects on tokamak equilibria using both single-fluid and two-fluid theory, with the principal emphasis on the latter. The two-fluid analysis has a number of distinctive features relating to the variation with respect to major radius R of various quantities on flux surfaces, and leads to non-trivial, experimentally-testable predictions. For example, when the ion and electron temperatures T_i and T_e are flux functions, and the ion flows in a given flux surface correspond to rigid body toroidal rotation, we have shown that two-fluid theory determines uniquely the rotation profile in terms of the temperature profiles (in MHD the rotation rate and temperature profile can be independently prescribed). Thus, by applying the analysis to measurements of temperature and rotation profiles one could test the assumption of rigid body rotation. With T_i and T_e assumed to be flux functions, we have shown that the two-fluid theory admits a far wider class of rotation, density and electrostatic potential profiles (varying non-trivially with respect to both poloidal flux Ψ and R) than those corresponding to rigid body rotation, which is required by ideal MHD in the absence of poloidal flows [cf. Eq. (13)]. Relaxing the assumption that ion temperature is a flux function leads to a still wider class of possible profiles.

We have also shown that ion momentum balance in the absence of ion poloidal flows leads to a generalised two-fluid Grad-Shafranov equation that is structurally similar to the standard ideal MHD form of this equation. We have computed leading order ion poloidal flow corrections to this equation, again casting it in a form that is closely analogous to the zero flow ideal MHD version. In principle, experimental profile data could be used to solve the two-fluid equations, thereby enabling the equilibrium structure to be determined more self-consistently than is possible in the framework of ideal MHD. In a future paper we intend to apply our two-fluid analysis to transonic MAST plasmas of the type discussed in Ref. [1].

Acknowledgments

We are grateful to Dr Jack Connor (UKAEA Culham Division), Ms Natalia Savenko (Uppsala University) and Professor Bryan Taylor (UKAEA Culham Division) for helpful discussions. This work was funded jointly by the United Kingdom Engineering and Physical Sciences Research Council and by the Commission of the European Communities under the Contract of Association between EURATOM and UKAEA. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

Appendix: Ion energy balance

The ion energy balance equation requires special consideration. In ideal MHD it is usual to assume that the plasma temperature is a flux function. Alternatively, one might assume, as in Ref. [15] for example, that the entropy is a flux function, with pressure and density being adiabatically related. However, when two charged species are present and $m_e \ll m_i$, neither assumption is necessarily appropriate under tokamak conditions. Strictly speaking, the ions are advected at their $\mathbf{E} \times \mathbf{B}$ drift velocity $\mathbf{V}_E$ rather than their fluid velocity $\mathbf{v}_i$ [20], and the ion energy balance equation has the following general structure (neglecting sources and transport processes, both turbulent and neoclassical, perpendicular to flux surfaces):

$$\frac{3}{2}n\mathbf{V}_E\cdot\nabla T_i + nT_i\nabla\cdot\mathbf{V}_E = -\nabla\cdot\mathbf{q}_\parallel^i, \tag{A1}$$

where

$$\mathbf{q}_\parallel^i = -n\chi_\parallel^i\frac{\mathbf{B}}{B}\nabla_\parallel T_i, \tag{A2}$$

is the parallel ion heat flux, $\chi_\parallel^i$ being an effective parallel ion thermal diffusivity (typically collisionless, and of order $Rv_{\rm th}^i$ where $v_{\rm th}^i$ is the ion thermal speed), and toroidal symmetry implies that we require only the components of $\mathbf{V}_E$ in the (R,Z) plane:

$$\mathbf{V}_E = \frac{f\nabla\phi\times\nabla\Phi}{B^2}, \qquad \text{(A3).}$$

These components of $\mathbf{V}_E$ can be large even when the ion poloidal fluid flow v_θ^i is negligible. The $\nabla\cdot\mathbf{V}_E$ term on the left hand side of Eq. (A1) vanishes in the infinite aspect ratio limit, and can thus be neglected to a first approximation under tokamak conditions. If $v_{\rm th}^i \gg |\mathbf{V}_E|$ it is clear that Eq. (A1) can be satisfied only if T_i is essentially a flux function. On the other hand, if the opposite inequality is satisfied Eq. (A1) reduces to

$$\frac{\partial(\Phi,T_i)}{\partial(R,Z)} \simeq 0, \qquad \text{(A4).}$$

To a first approximation, T_i would then be a function of Φ (and hence of n) and vary on a flux surface. If, as in MAST plasmas with counter-current neutral beam injection, the toroidal flows are transonic [1], the electric field must, by definition, produce drifts of the order of $v_{\rm th}^i$. Equation (A1) suggests that significant variations of T_i on a flux surface could occur under these circumstances. For this reason, although in this paper we have considered in detail the case of T_i being a flux function, it is more appropriate for plasmas with strong toroidal flows to specify $p_{\rm tot}(\Psi, R^2)$ in terms of the profile functions $P^*(\Psi)$, $\sigma_j(\Psi)$ and then to evaluate the rotation $[\omega_{-j}(\Psi)]$ and ion temperature $[\theta_{-j}(\Psi)]$ profile functions self-consistently. In this approach the ion energy equation is not required. Of course it is possible, in principle, to solve the ion energy equation with appropriate sources and transport coefficients, along with the other equations of the two-fluid system, in order to to determine the equilibrium. This, however, is almost as challenging as determining the entire set of flux functions using transport modelling. The analysis presented in the present paper, when used in conjunction with experimental measurements (of rotation or density, for example), provides a more practical method of determining the ion temperature distribution.

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