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Plasma physics in non-inertial frames

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Abstract

Equations describing the non-relativistic motion of a charged particle in an arbitrary non-inertial reference frame are derived from the relativistically-invariant form of the particle action. It is shown that the equations of motion can be written in the same form in inertial and non-inertial frames, with the effective electric and magnetic fields in the latter modified by inertial effects associated with centrifugal and Coriolis accelerations. These modifications depend on the particle charge-to-mass ratio, and also the vorticity, specific kinetic energy and compressibility of the frame flow. The Vlasov, Fokker-Planck, gyrokinetic and fluid equations in such a frame are derived. The results are applied to tokamak plasmas rotating about the machine symmetry axis with a non-relativistic but otherwise arbitrary toroidal flow velocity. Astrophysical applications of the analysis are also possible, since the power of the action principle is such that it can be used to describe relativistic flows in curved spacetime.

I. Introduction

Historically, the equations of plasma physics have generally been studied using inertial reference frames, not least because Maxwell's equations have their simplest form in such frames. However, tokamak plasmas often rotate at velocities comparable to or exceeding the thermal velocity of at least one of the plasma ion species [1], while much higher rotation rates are encountered in some astrophysical plasmas, such as pulsar magnetospheres [2]. It is likely that the observed behaviour of such systems can be better understood in some respects by considering the equations determining their evolution in suitable non-inertial frames. In this paper a first-principles approach is used to formulate the Newton-Lorentz, Vlasov, Fokker-Planck and plasma fluid equations in an arbitrary accelerating frame, with particular emphasis on the case of a rotating frame.

In the tokamak context work by Brizard [3] on the extension of gyrokinetic theory to rotating frames has recently been applied by Peeters and co-workers [4, 5, 6] to the study of turbulent particle, heat and momentum transport. In this paper we propose an alternative to the approach used by Brizard by taking into account the exact form in the rotating frame of the Newton-Lorentz equations and the associated Vlasov and Fokker-Planck equations, from which the gyrokinetic equation is derived. Through the use of a relativistically-invariant action principle, the exact equations are simpler to manipulate than the gyro-averaged equations. Once the exact equations in the non-inertial frame are established, it is relatively straightforward to construct the appropriate drift orbit theory, and the gyrokinetic formalism can then be readily applied using the Whitham "averaged Lagrangian" method [7, 8], and methods developed by Bernstein and Catto [9] to obtain nonlinear gyrokinetic equations.

The paper is organized as follows. In the next section we discuss an elementary kinematic approach to the description of charged particle motion in a uniformly-rotating frame with constant angular velocity. In this simple case the Newton-Lorentz equations can be written in exactly the same form as in the laboratory (inertial) frame by introducing equivalent electric and magnetic fields. In Sec. III nonrelativistic charged particle motion in an arbitrary accelerating frame is considered, using Hamilton's principle and a relativistically-invariant form of the single-particle Lagrangian. We show that by introducing equivalent scalar and vector potentials the Newton-Lorentz equations in the accelerating and inertial frames can be made formally identical. The Lagrangian, Hamiltonian and canonical momenta of the particles are discussed and conservation laws explained. In Sec. IV we use the Hamiltonian in the accelerating frame to obtain the Vlasov equation for the particle distribution function in that frame. The corresponding drift orbit equations are derived by using a standard averaged Lagrangian argument, applied to equivalent fields rather than inertial frame fields. The corresponding gyrokinetic and moment (fluid) equations are obtained in the dissipationless limit. In Sec. V we consider applications of the analysis to the Joint European Torus (JET) [1], a conventional aspect ratio tokamak with relatively moderate rotation, and the Mega-Ampère Spherical Tokamak (MAST) [10], a tight aspect ratio device in which the effects of rotation on the equivalent fields are somewhat greater than they are in JET. We also discuss how the frame flow can be related to the electrostatic potential in the laboratory frame when the flow is due to rotation of the bulk ion species. In Sec. VI we compare and contrast our results with previous work in this area. In Sec. VII we present a summary of our results and conclusions.

II. Charged particle motion in uniformly-rotating frames

We study in the first instance a uniformly-rotating reference frame with a fixed rotation axis and constant angular velocity $\mathbf{\Omega} = \Omega \mathbf{e}_z$, where $\mathbf{e}_z$ is the unit vector in the z-direction. The laboratory frame $\mathbf{K}_{\rm lab}$ is taken to be an inertial frame of reference with a fixed origin and the rotating frame is denoted by $\mathbf{K}_{\rm rot}$. We consider the non-relativistic limit and thus assume that all relevant lengths in the problem are much smaller than c/Ω (the light cylinder radius). If the two frames coincide at $t = 0$, a point P at $\mathbf{r} = (x, y, z)$ in $\mathbf{K}_{\rm lab}$ has rotating frame coordinates $\mathbf{R} = (X, Y, Z)$ at time t which satisfy the equations

$$x = X \cos \Omega t - Y \sin \Omega t, \tag{1}$$

$$y = X \sin \Omega t + Y \cos \Omega t, \tag{2}$$

$$z = Z. \tag{3}$$

It is well-known (see e.g. [11]) that rates of change of position in $\mathbf{K}_{\rm rot}$ and $\mathbf{K}_{\rm lab}$ are related by the expression

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{R}}{dt} + \mathbf{\Omega} \times \mathbf{R}, \tag{4}$$

where the time derivatives are taken in each frame with the corresponding basis vectors held constant. Denoting the laboratory and rotating frame time derivatives by d/dt and $\partial/\partial t$ respectively, we obtain from Eq. (4)

$$\begin{aligned} \frac{d^2\mathbf{r}}{dt^2} &= \left[\frac{\partial}{\partial t} + \mathbf{\Omega}\times\right]\left[\frac{\partial \mathbf{R}}{\partial t} + \mathbf{\Omega} \times \mathbf{R}\right] \\ &= \frac{\partial^2\mathbf{R}}{\partial t^2} + 2\mathbf{\Omega}\times\left(\frac{\partial \mathbf{R}}{\partial t}\right) + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{R}) \\ &= \frac{\partial^2\mathbf{R}}{\partial t^2} + 2\mathbf{\Omega}\times\left(\frac{\partial \mathbf{R}}{\partial t}\right) - \frac{1}{2}\nabla\left(\Omega^2 R^2\right), \end{aligned} \tag{5}$$

where $R = (X^2 + Y^2)^{1/2} = (x^2 + y^2)^{1/2}$. Hence, for a particle of mass m Newton's second law in the rotating frame takes the form

$$m\frac{\partial^2\mathbf{R}}{\partial t^2} + 2m\mathbf{\Omega} \times \left(\frac{\partial \mathbf{R}}{\partial t}\right) - \frac{m}{2}\nabla\left(\Omega^2 R^2\right) = \mathbf{F}, \tag{6}$$

where the force on the particle $\mathbf{F}$ is frame-independent but its components may depend on Ω if they are written in terms of rotating frame coordinates. We note from Eq. (6) the well-known result that two pseudo-forces appear in the equations of motion in the rotating frame: the Coriolis force, which is velocity dependent but does no work, equal to $2m(\partial\mathbf{R}/\partial t) \times \mathbf{\Omega}$, and the centrifugal force, which can be written as the gradient of a potential equal to $m\Omega^2R^2/2$.

In the non-relativistic limit (i.e. neglecting terms of order Ω^2R^2/c^2 where c is the speed of light) the electric and magnetic fields in the laboratory and rotating frames are related by the expressions

$$\mathbf{B}_{\text{rot}} = \mathbf{B}_{\text{lab}}, \tag{7}$$

$$\mathbf{E}_{\text{rot}} = \mathbf{E}_{\text{lab}} + (\mathbf{\Omega} \times \mathbf{R}) \times \mathbf{B}_{\text{lab}}, \tag{8}$$

and the Newton-Lorentz equations in these frames for particles of charge Ze (e being the proton charge) are

$$m\frac{d\mathbf{v}}{dt} = Ze\left(\mathbf{E}_{\text{lab}} + \mathbf{v} \times \mathbf{B}_{\text{lab}}\right), \tag{9}$$

$$m\frac{\partial \mathbf{V}}{\partial t} = Ze\left(\mathbf{E}_{\text{rot}} + \mathbf{V} \times \mathbf{B}_{\text{rot}}\right) + 2m\mathbf{V} \times \mathbf{\Omega} + \frac{m}{2}\nabla\left(\Omega^2R^2\right), \tag{10}$$

where $\mathbf{V} = \partial\mathbf{R}/\partial t = \mathbf{v} - \mathbf{\Omega} \times \mathbf{R}$, $\mathbf{v}$ being the particle's velocity in the laboratory frame. Equation (10) can be written in the form

$$m\frac{\partial \mathbf{V}}{\partial t} = Ze\left[\mathbf{E}_{\text{rot}} + \frac{m}{Ze}\nabla\left(\frac{\Omega^2R^2}{2}\right) + \mathbf{V} \times \left(\mathbf{B} + \frac{2m}{Ze}\mathbf{\Omega}\right)\right]. \tag{11}$$

If the fields are time-independent in the rotating frame we may write $\mathbf{E}_{\rm rot} = -\nabla\Phi_{\rm rot}$ and define equivalent electric and magnetic fields

$$\mathbf{E}_* = -\nabla\Phi_*, \tag{12}$$

$$\mathbf{B}_* = \mathbf{B} + \frac{2m}{Ze}\boldsymbol{\Omega}, \tag{13}$$

where the equivalent electric potential is defined by the expression

$$\Phi_* = \Phi_{\rm rot} - \frac{m}{2Ze}\Omega^2 R^2. \tag{14}$$

The Newton-Lorentz equations in the rotating frame then have the familiar form

$$m\frac{\partial \mathbf{V}}{\partial t} = Ze\left(\mathbf{E}_* + \mathbf{V}\times\mathbf{B}_*\right). \tag{15}$$

It is apparent from Eq. (15) that in a uniformly-rotating frame with constant electric and magnetic fields a charged particle moves exactly as it would do in an inertial frame in the presence of electric and magnetic fields $\mathbf{E}_*$ and $\mathbf{B}_*$ which will, in general, differ in both magnitude and direction from the fields $\mathbf{E}_{\rm rot}$ and $\mathbf{B}_{\rm rot}$. In particular the particle gyrates about $\mathbf{B}_*$ rather than $\mathbf{B}_{\rm rot}$ and, if the usual drift ordering applies, undergoes grad-B, curvature and $\mathbf{E}\times\mathbf{B}$ drifts that are determined by $\mathbf{E}_*$, $\mathbf{B}_*$ rather than $\mathbf{E}_{\rm rot}$, $\mathbf{B}_{\rm rot}$ or $\mathbf{E}_{\rm lab}$, $\mathbf{B}_{\rm lab}$. Apart from the assumptions of non-relativistic motions and constant uniform $\boldsymbol{\Omega}$, this result is completely general and hence valid for arbitrary values of Ω, m, Ze, $\mathbf{E}_{\rm lab}$ and $\mathbf{B}_{\rm lab}$. Every gyro-averaged orbit or kinetic equation in the rotating frame must be consistent with Eq. (15).

The effective electrostatic potential Φ_* is modified by a centrifugal term, the associated force being directed radially outward from the rotation axis. The centrifugal potential may be written as

$$\Phi_{\rm cent} = -\frac{T}{Ze}\frac{\Omega^2 R^2}{V_{\rm th}^2} \equiv -\frac{T}{Ze}M_\Omega^2,$$

where $T \equiv mV_{\rm th}^2/2$. If T is taken to be the temperature of a species with mass m and charge Ze whose mean flow velocity coincides with that of the rotating frame, M_Ω is then the sonic Mach number of the flow. The centrifugal potential is independent of the rotation direction and depends only upon the Mach number and the temperature.

As noted recently by McClements and McKay [12], the Coriolis force simply adds to the stationary magnetic field a uniform component parallel to the rotation axis. This is an immediate consequence of Larmor's theorem, which states that the behavior of a charged particle of charge-to-mass ratio Ze/m in a uniform magnetic field $\mathbf{B}$ is indistinguishable from its behavior in a frame rotating uniformly at a rate $\boldsymbol{\Omega} = (Ze/2m)\mathbf{B}$ in the absence of a magnetic field [13]. The strength of the uniform field relative to the stationary field is measured by a dimensionless rotation parameter $\rho_\Omega^* \equiv 2\Omega/\omega_c$ where $\omega_c = ZeB/m$ is the cyclotron frequency of the particle in the magnetic field $\mathbf{B}_{\rm rot}$. The vertical magnetic field due to the Coriolis force in the rotating frame is proportional to Ω and hence, unlike the centrifugal force, depends upon the sign of the rotation.

Henceforth we shall (unless stated otherwise) work in the rotating frame with $\mathbf{\Omega} = \Omega\mathbf{e}_z$ and write $\partial/\partial t = d/dt$. Charged particles execute Larmor gyrations about $\mathbf{B}_*$ with effective gyro frequency

$$\omega_c^* = \frac{Ze}{m} B_*, \tag{16}$$

and effective Larmor radius

$$r_{\rm L}^* = \frac{c_\perp}{\omega_c^*}. \tag{17}$$

The effective magnetic field has magnitude

$$B_* = B\left[1 + 2\rho_\Omega^* \mathbf{b}\cdot\mathbf{e}_Z + (\rho_\Omega^*)^2\right]^{1/2}, \tag{18}$$

where $\mathbf{b} = \mathbf{B}/B$ and $c_\perp$ is the particle's gyration velocity about $\mathbf{B}_* = B_*\mathbf{b}_*$. Subject to the usual drift ordering requirements on the equivalent fields, adiabatic invariance applies to collisionless orbits, the effective magnetic moment being $\mu^* = mc_\perp^2/(2B_*) \equiv \mathcal{E}_\perp^*/B_*$. Moreover toroidal canonical momentum will be conserved if the equivalent fields are axisymmetric about the rotation axis, and there will be a conserved energy integral of the particle motion if the equivalent fields are time independent.

III. Charged particle motion in general non-inertial frames

In the limit in which the emission and absorption of radiation can be neglected, the action for a particle of rest mass m and charge Ze in an electromagnetic field can be written in the generally covariant form [13]

$$S = -\int_{\tau_{\rm in}}^{\tau_{\rm fin}} [mv_\mu v^\mu + ZeA_\mu v^\mu] d\tau. \tag{19}$$

Here $\tau_{\rm in}$ and $\tau_{\rm fin}$ are the initial and final proper time, $v^\mu = dx^\mu/d\tau$ where x^μ is an arbitrary set of spacetime coordinates, and A_μ is the covariant form of the electromagnetic four-potential. In Minkowski spacetime the proper time interval $d\tau$ is related to the coordinate time interval dt by the relation $\gamma d\tau = dt$, $\gamma = [1 - v^2/c^2]^{-1/2}$ being the usual Lorentz factor for a particle with three-velocity $\mathbf{v} = d\mathbf{r}/dt$. In this spacetime the contravariant and covariant forms of the particle four-velocity are respectively $v^\mu = \gamma(c, \mathbf{v})$ and $v_\mu = \gamma(c, -\mathbf{v})$, while $A_\mu = (\Phi/c, -\mathbf{A})$ where $\mathbf{A}$ is the magnetic vector potential. Equation (19) may then be written in the equivalent form

$$S = \int_{t_{\rm in}}^{t_{\rm fin}} \mathcal{L} dt, \tag{20}$$

where $t_{\rm in}$, $t_{\rm fin}$ are now the initial and final coordinate time and

$$\mathcal{L} = -mc^2(1 - v^2/c^2)^{1/2} + Ze\left(\mathbf{v}\cdot\mathbf{A} - \Phi\right), \tag{21}$$

is the Lagrangian. Invoking Hamilton's principle of least action, i.e. requiring that the trajectory of the particle between $t_{\rm in}$ and $t_{\rm fin}$ is such that S has a stationary value, leads to the well-known Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial v_i}\right) = \frac{\partial\mathcal{L}}{\partial x_i}, \tag{22}$$

where x_i denotes a general spatial coordinate of the particle and $v_i = dx_i/dt$. The key point here is that Eq. (22) is completely general for a particle in Minkowski spacetime that is not emitting or absorbing radiation, in that it yields the equations of motion in any frame, inertial or otherwise.

In the following $\mathbf{u}_f = \mathbf{v} - \mathbf{V}$ denotes a non-relativistic flow in $\mathbf{K}_{\rm lab}$, varying in space and time; the magnetic vector and electric potentials in this frame are denoted by $\mathbf{A}_{\rm lab}$ and $\Phi_{\rm lab}$. It follows that $\mathbf{E}_{\rm lab} = -\partial\mathbf{A}_{\rm lab}/\partial t - \nabla\Phi_{\rm lab}$ and $\mathbf{B}_{\rm lab} = \nabla\times\mathbf{A}_{\rm lab}$. The potentials in the laboratory frame are related to those in the flow frame $\mathbf{A}_f$, Φ_f by the expressions [13]

$$\mathbf{A}_{\rm lab} = \mathbf{A}_f, \tag{23}$$

$$\Phi_{\rm lab} - \mathbf{A}_{\rm lab}\cdot\mathbf{u}_f = \Phi_f. \tag{24}$$

Equation (24) is a consequence of the invariance of $\mathbf{A}\cdot\mathbf{v} - \Phi$, i.e.

$$\mathbf{A}_{\rm lab}\cdot\mathbf{v} - \Phi_{\rm lab} = \mathbf{A}_f\cdot\mathbf{V} - \Phi_f, \tag{25}$$

which arises, in the non-relativistic limit, from the manifest invariance of $A_\mu v^\mu$. Equations (23-24) are equivalent to the well-known formulae for the transformation of electric and magnetic fields in the non-relativistic limit [cf. Eqs. (7-8)]:

$$\mathbf{B}_f = \mathbf{B}_{\rm lab}, \tag{26}$$

$$\mathbf{E}_f = \mathbf{E}_{\rm lab} + \mathbf{u}_f\times\mathbf{B}_{\rm lab}. \tag{27}$$

It is evident from Eq. (21) that $\gamma\mathcal{L}$ is a Lorentz invariant, and therefore, in the non-relativistic limit that we are considering, $\mathcal{L}$ is invariant under transformations between $\mathbf{K}_{\rm lab}$ and the flow frame, $\mathbf{K}_f$. Putting $\mathbf{v} = \mathbf{V} + \mathbf{u}_f$, $\mathbf{A} = \mathbf{A}_{\rm lab} = \mathbf{A}_f$ and $\Phi = \Phi_{\rm lab} = \Phi_{\rm f} + \mathbf{A}_f\cdot\mathbf{u}_f$ in Eq. (21), taking the non-relativistic limit, and neglecting the rest mass term mc^2, we obtain (setting $u_f^2 = \mathbf{u}_f.\mathbf{u}_f$)

$$\mathcal{L} = \frac{1}{2}mV^2 + Ze\mathbf{V}\cdot\left[\mathbf{A}_f + \frac{m}{Ze}\mathbf{u}_f\right] - Ze\left[\Phi_f - \frac{mu_f^2}{2Ze}\right]. \tag{28}$$

Dropping the suffix on $\mathbf{A}$, we note that the canonical momenta in the flow frame are given by

$$\mathbf{P} = \frac{\partial\mathcal{L}}{\partial\mathbf{V}} = m\mathbf{V} + Ze\left[\mathbf{A} + \frac{m}{Ze}\mathbf{u}_f\right]. \tag{29}$$

The Hamiltonian in the flow frame $\mathcal{H} \equiv \mathbf{P}\cdot\mathbf{V} - \mathcal{L}$ is then given by

$$\mathcal{H} = \frac{1}{2m}\left[\mathbf{P} - Ze\left(\mathbf{A} + \frac{m}{Ze}\mathbf{u}_f\right)\right]^2 + Ze\left[\Phi_f - \frac{mu_f^2}{2Ze}\right]. \tag{30}$$

Introducing two equivalent potentials,

$$\mathbf{A}_* = \mathbf{A} + \frac{m}{Ze}\mathbf{u}_f \tag{31}$$

$$\begin{aligned}\Phi_* &= \Phi_f - \frac{mu_f^2}{2Ze} \\ &= \Phi_{\rm lab} - \mathbf{A}\cdot\mathbf{u}_f - \frac{mu_f^2}{2Ze},\end{aligned} \tag{32}$$

we note that the Lagrangian and Hamiltonian can be written in the simple form

$$\mathcal{L} = \frac{1}{2}mV^2 + Ze\mathbf{V}\cdot\mathbf{A}_* - Ze\Phi_*, \tag{33}$$

$$\mathcal{H} = \frac{1}{2m}\left(\mathbf{P} - Ze\mathbf{A}_*\right)^2 + Z_e\Phi_*. \tag{34}$$

In terms of the equivalent potentials, the Euler-Lagrange equations are

$$m\dot{\mathbf{V}} + Ze\frac{\partial \mathbf{A}_*}{\partial t} + Ze\left(\mathbf{V}\cdot\nabla\right)\mathbf{A}_* = Ze\nabla\left(\mathbf{V}\cdot\mathbf{A}_*\right) - Ze\nabla\Phi_*, \tag{35}$$

where ∇ is the gradient operator in the flow frame and we have used the fact that the required time derivative of $\mathbf{A}_*$ is the rate of change of this vector along the particle trajectory:

$$\frac{d\mathbf{A}_*}{dt} = \frac{\partial \mathbf{A}_*}{\partial t} + \left(\mathbf{V}\cdot\nabla\right)\mathbf{A}_*. \tag{36}$$

The particle velocity in the flow frame $\mathbf{V}$ can be treated as a constant vector as far as the ∇ operator is concerned, and therefore the following identity holds:

$$\nabla\left(\mathbf{V}\cdot\mathbf{A}_*\right) = \left(\mathbf{V}\cdot\nabla\right)\mathbf{A}_* + \mathbf{V}\times\left(\nabla\times\mathbf{A}_*\right).$$

Hence the Euler-Lagrange equations can be written in the form

$$m\frac{d\mathbf{V}}{dt} = Ze\left(\mathbf{E}_* + \mathbf{V}\times\mathbf{B}_*\right), \tag{37}$$

where the equivalent fields are given by

$$\begin{aligned}\mathbf{E}_* &= -\nabla\Phi_* - \frac{\partial \mathbf{A}_*}{\partial t} \\ &= \mathbf{E}_f - \frac{m}{Ze}\frac{\partial \mathbf{u}_f}{\partial t} + \frac{m}{2Ze}\nabla u_f^2,\end{aligned} \tag{38}$$

$$\begin{aligned}\mathbf{B}_* &= \nabla\times\mathbf{A}_* \\ &= \mathbf{B}_f + \frac{m}{Ze}\mathbf{W}_f.\end{aligned} \tag{39}$$

Here, $\mathbf{W}_f = \nabla\times\mathbf{u}_f$ is the vorticity of the frame flow, and $\mathbf{E}_f$, $\mathbf{B}_f$ are the electric and magnetic fields in $\mathbf{K}_f$ at the location of the particle. Putting $\mathbf{u}_f = \mathbf{\Omega}\times\mathbf{R}$ where $\mathbf{\Omega} = \Omega\nabla Z$ is a uniform and constant angular velocity, and assuming time-independent fields in the flow frame, we recover Eqs. (12-15) from the expressions given above; in this case $\mathbf{W}_f = 2\mathbf{\Omega}$. If Ω depends on R and Z, which is normally the case in rotating tokamak plasmas, $\mathbf{W}_f$ has both radial and vertical components.

Equation (37) indicates that the particle gyrates around $\mathbf{B}_*$ rather than $\mathbf{B}_{\rm lab}$ or $\mathbf{B}_f$ and its drifts are determined by $\mathbf{E}_*$, $\mathbf{B}_*$ rather than $\mathbf{E}_{\rm lab}$, $\mathbf{B}_{\rm lab}$ (or $\mathbf{E}_f$, $\mathbf{B}_f$). This distinction between fields in inertial and non-inertial frames is exact and must be respected in any correct formulation of the charged particle dynamics. The effective fields depend upon $\mathbf{u}_f$ and m/Ze, but not on the particle's velocity in either the laboratory frame or the flow frame. Hence the motions of all particles with the same charge-to-mass ratio will be determined by the same effective fields.

The physical meaning of these equations should be clear: the Coriolis-like inertial effects are velocity-dependent but can do no work on the particle. Hence they *must* be perpendicular to the velocity vector of the particle in the frame and must therefore combine with the magnetic field in the Lorentz-Newton equations to result in $\mathbf{B}_*$. The centrifugal-like inertial effects are non-dissipative and in general not parallel to the equivalent field. They do not depend upon the velocity vector of the particle either. However, unlike the Coriolis effect, they do depend upon the gradients of the specific kinetic energy of the frame flow. Hence they produce purely conservative accelerations along and perpendicular to the equivalent magnetic field, and thus contribute to the equivalent electrostatic potential as described by Eq.(32).

The relative size of the modification to the effective magnetic field scales as W_f/ω_c. The effective electric field depends on a centrifugal potential $m/(2Ze)u_f^2$ and also contains a term proportional to $\partial \mathbf{u}_f/\partial t$. Since the electric potential in a frame co-moving with the plasma is typically of the order of T_e/e where T_e is electron temperature and $-e$ is electron charge, the ratio of the centrifugal potential to the electric potential in the co-moving frame is around $(m/m_i)M_f^2/Z$, where $M_f = u_f/v_i$ is the Mach number of the bulk ion flow, m_i being the bulk ion mass and $v_i = (2T_e/m_i)^{1/2}$ the bulk ion thermal speed.

In terms of cylindrical coordinates (R, φ, Z) in $\mathbf{K}_f$ the Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}m\left(\dot{R}^2 + R^2\dot{\varphi}^2 + \dot{Z}^2\right) + Ze\left(\dot{R}A_{*R} + \dot{\varphi}RA_{*\varphi} + \dot{Z}A_{*Z}\right) - Ze\Phi_*. \qquad (40)$$

The corresponding canonical momenta and Hamiltonian are

$$P_R = m\dot{R} + Z_I eA_{*R}, \qquad (41)$$

$$P_\varphi = mR^2\dot{\varphi} + ZeRA_{*\varphi}, \qquad (42)$$

$$P_Z = m\dot{Z} + ZeA_{*Z} \qquad (43)$$

$$\mathcal{H} = \frac{1}{2m}\left[(P_R - ZeA_{*R})^2 + (P_\varphi - ZeA_{*\varphi})^2 + (P_Z - Z_eA_{*Z})^2\right] + Ze\Phi_*. \qquad (44)$$

If Φ_* and $\mathbf{A}_*$ are symmetric about the Z-axis $\mathcal{L}$ and $\mathcal{H}$ are independent of φ. In these circumstances it follows from Hamilton's equations that P_φ is a constant of the motion. It should be noted, however, that this conserved quantity is defined by the effective vector potential given by Eq. (31) rather than the laboratory frame vector potential. If moreover Φ_* and $\mathbf{A}_*$ are time-independent, there is a conserved energy integral of the particle motion in the flow frame (namely, $\mathcal{H}$ itself).

If $\mathbf{u}_f$ is a purely toroidal azimuthally-symmetric flow it is divergence-free and only modifies the poloidal magnetic field. If the rotation is in the co-current direction, it reduces the effective poloidal field outboard of the magnetic axis and increases it on the inboard side. As noted in Ref. [12], this effect has consequences for the orbits and hence neoclassical transport of massive impurity ions. If $\mathbf{u}_f$ has a spatially-varying poloidal component, the effective toroidal magnetic field is also modified.

IV. Kinetic and fluid equations in non-inertial frames

A number of important consequences follow from the Lagrangian and Hamiltonian derived in the preceding section. In the absence of collisions the particle distribution function $f(q_k, P_k, t)$ in the flow frame satisfies the single-particle Liouville equation

$$\frac{\partial f}{\partial t} + \sum_{k=1}^{3} \left[\frac{\partial \mathcal{H}}{\partial P_k} \frac{\partial f}{\partial q_k} - \frac{\partial \mathcal{H}}{\partial q_k} \frac{\partial f}{\partial P_k} \right] = 0, \tag{45}$$

where P_k are the canonical momenta associated with general coordinates q_k. Using the expression for $\mathcal{H}$ given by Eq. (34) together with Hamilton's equations, we find that Eq. (45) reduces to the standard form of the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{V} \cdot \nabla f + \frac{Ze}{m} \left(\mathbf{E}_* + \mathbf{V} \times \mathbf{B}_* \right) \cdot \frac{\partial f}{\partial \mathbf{V}} = 0, \tag{46}$$

where, as before, ∇ is the gradient operator in $\mathbf{K}_f$. Equation (46) provides an exact description of dissipationless plasma behavior in the non-inertial frame: if the appropriate orderings apply, one may use this equation or Eq. (45) as the basis for obtaining reduced kinetic and fluid descriptions of the plasma, respectively by averaging over particle orbits and by calculating velocity-space moments. It is important to note that these procedures should only be carried out once the exact kinetic equation in the non-inertial frame has been obtained, rather than vice versa.

The derivation of drift orbit theory and the associated gyrokinetic theory in the frame $\mathbf{K}_f$ can proceed, subject to usual orderings, by computing a gyro-averaged Lagrangian from the exact expression given by Eq. (33), using an approach developed by Whitham [8, 7]. One may then obtain a gyrokinetic Hamiltonian, with the adiabatically-invariant magnetic moment μ^* (defined in terms of gyro-motion around the effective magnetic field, $\mathbf{B}_*$) as one of the canonical momenta. The collisionless gyrokinetic equation follows from the corresponding Liouville equation.

Since Eq. (37) is in the same form as the Newton-Lorentz equation in an inertial frame, we can write down the corresponding guiding center equation, which is applicable if the usual drift ordering holds with respect to the effective fields [8]:

$$m \frac{d\mathbf{V}^*_{\mathrm{gc}}}{dt} = ZeB_* \mathbf{V}^*_{\mathrm{gc}} \times \mathbf{b}_* + Ze\mathbf{E}_* - \mu^* \nabla B_*, \tag{47}$$

where $\mathbf{V}_{\mathrm{gc}}$ is the particle guiding center velocity and $\mathbf{b}_* = \mathbf{B}_*/B_*$. Equation (47) has formal solution

$$\mathbf{V}^*_{\mathrm{gc}} = V^*_{\|} \mathbf{b}_* + \mathbf{v}_D, \tag{48}$$

where $V_\parallel^* = \mathbf{V}_{\rm gc} \cdot \mathbf{b}_*$ and

$$\mathbf{v}_D \;=\; \left(\frac{\mathbf{E}_*}{B_*} - \frac{\mu^* \nabla B_*}{ZeB_*} - \frac{1}{\omega_c} \frac{d\mathbf{V}_{\rm gc}^*}{dt} \right) \times \mathbf{b}_*. \tag{49}$$

The collisionless drift-kinetic equation then takes the form [14]

$$\frac{\partial F}{\partial t} + \left(V_\parallel^* \mathbf{b}_* + \mathbf{v}_D \right) \cdot \nabla F + \left[Ze \frac{\partial \Phi_*}{\partial t} + \mu_* \frac{\partial B_*}{\partial t} - ZeV_\parallel^* \mathbf{b}_* \cdot \frac{\partial \mathbf{A}_*}{\partial t} \right] \frac{\partial F}{\partial \mathcal{E}} \;=\; 0, \tag{50}$$

where F is the guiding center distribution and

$$\mathcal{E} \;=\; \frac{1}{2} m \left(V_\parallel^* \right)^2 + Ze\Phi_* + \mu_* B_*. \tag{51}$$

is the particle energy in $\mathbf{K}_f$. The more exact gyrokinetic equation (with or without collisions) in $\mathbf{K}_f$ can be obtained using methods developed by Bernstein and Catto [9], with the sole modification that $\mathbf{E}_*$, $\mathbf{B}_*$ and Φ_* must be used rather than the Maxwell fields and potentials. In the notation used by Bernstein and Catto, the non-linear, electromagnetic, collisional, gyrokinetic equation takes the form

$$\frac{\partial F}{\partial t} + \langle\langle \dot{\mathbf{r}}' \rangle\rangle \cdot \nabla' F + \langle\langle \dot{u}' \rangle\rangle \frac{\partial F}{\partial u'} \;=\; \langle\langle C \rangle\rangle, \tag{52}$$

where $\langle\langle ... \rangle\rangle$ denotes a gyrokinetic average, $\mathbf{r}'$, u' respectively denote guiding center position and parallel velocity, and C is the Fokker-Planck collision term. The calculation of $\langle\langle C \rangle\rangle$ is identical in $\mathbf{K}_f$ and $\mathbf{K}_{\rm lab}$.

Collisions (and other dissipative effects, such as radiation) can be readily incorporated into the Newton-Lorentz equations in the non-inertial frame by adding drag and Langevin force terms [15]. For the case of trace impurity ions colliding with bulk ions the equations can be written in the form

$$m \frac{d\mathbf{V}}{dt} \;=\; Ze \left(\mathbf{E}_* + \mathbf{V} \times \mathbf{B}_* \right) + \frac{m}{\tau} (\mathbf{u}_i - \mathbf{V}) + \mathbf{f}, \tag{53}$$

where $\mathbf{u}_i$ is the bulk ion fluid flow in $\mathbf{K}_f$, τ is the momentum relaxation rate for the impurity species due to collisions with bulk ions, and $\mathbf{f}$ is a random stochastic force. This equation has recently been used in the laboratory frame to investigate collisional trace impurity transport in MAST [12, 16]. Collisions with other species (in particular electrons) can be easily included in Eq. (53).

One may also derive fluid equations in $\mathbf{K}_f$ by taking moments of the Vlasov or Fokker-Planck equations. In the simplest approximation, with isotropic pressure and no dissipation, the continuity, momentum and energy equations are

$$\frac{\partial n_Z}{\partial t} + \nabla \cdot (n_Z \mathbf{v}_Z) \;=\; 0, \tag{54}$$

$$m n_Z \left(\frac{\partial \mathbf{v}_Z}{\partial t} + (\mathbf{v}_Z \cdot \nabla) \mathbf{v}_Z \right) \;=\; -\nabla p_I + Zen_Z \left(\mathbf{E}_* + \mathbf{v}_Z \times \mathbf{B}_* \right), \tag{55}$$

$$\frac{3}{2}n_Z\left(\frac{\partial T_Z}{\partial t}+(\mathbf{v}_Z\cdot\nabla)\,T_Z\right)+n_ZT_Z\nabla\cdot\mathbf{v}_Z, \quad = \quad 0 \tag{56}$$

where n_Z, $\mathbf{v}_Z$, T_Z and $p_Z = n_Z T_Z$ are, respectively, the number density, flow velocity, temperature and pressure of the species under consideration in the non-inertial frame. Under steady-state conditions these equations reduce to

$$\nabla\cdot(n_Z\mathbf{v}_Z) \quad = \quad 0, \tag{57}$$

$$mn_Z\mathbf{W}\times\mathbf{v}_Z \quad = \quad -\nabla p_Z-\frac{1}{2}mn_Z\nabla v_Z^2+Zen_Z\left(-\nabla\Phi_*+\mathbf{v}_Z\times\mathbf{B}_*\right), \tag{58}$$

$$\frac{3}{2}n_Z\left(\mathbf{v}_Z\cdot\nabla\right)T_Z+n_ZT_Z\nabla\cdot\mathbf{v}_Z \quad = \quad 0, \tag{59}$$

where $\mathbf{W} = \nabla\times\mathbf{v}_Z$. The inertia terms in Eq. (58) can be combined with the electromagnetic force terms to yield the simple pressure balance relation

$$\frac{1}{Zen_Z}\nabla p_Z \quad = \quad -\nabla\hat{\Phi}+\mathbf{v}_Z\times\hat{\mathbf{B}}, \tag{60}$$

where $\hat{\Phi} = \Phi_* + mv_Z^2/(2Ze)$ and $\hat{\mathbf{B}} = \mathbf{B}_* + (m/Ze)\mathbf{W}$.

One can extend the above analysis in a straightforward manner to include dissipation and sources of particles, momentum and energy. All of the plasma species can be treated in the same way, although it should be noted that the effective fields depend on the charge-to-mass ratio of the species in question. Thus, the standard equations of plasma physics, both kinetic and fluid, can be transformed rather simply to non-relativistic but otherwise arbitrary co-moving frames. The Maxwell fields can be obtained in the laboratory frame, since the transformation of these fields to the co-moving frame is straightforward.

V. Applications to tokamaks

The relevance of the above analysis for tokamak plasmas can be assessed by making some simple estimates. In JET, for example, typical values of the toroidal magnetic field, plasma current and bulk ion (deuterium) temperature are, respectively, $B \simeq 3\,\mathrm{T}$, $I_p \simeq 3\,\mathrm{MA}$ and $T_i \simeq 10\,\mathrm{keV}$, and the toroidal rotation rate can be up to around $2\times 10^5\,\mathrm{rads}^{-1}$, with rotation velocities of up to about $1000\,\mathrm{kms}^{-1}$ [1]. For deuterons these figures indicate that the effective vertical magnetic field associated with the Coriolis force $2m\Omega/(Ze)$ appearing in our analysis is less than about $8\,\mathrm{mT}$; typical values of the poloidal magnetic field in JET are almost two orders of magnitude larger than this, and therefore any additional drift orbit effects arising from this Coriolis force are likely to be negligible. On the other hand the sonic Mach number of the bulk ion flow M_f can be of order unity, and therefore in the co-moving frame the centrifugal potential can be comparable in magnitude to the electric potential. Thus, the effects of rotation in JET on charged particle orbits with charge-to-mass ratios similar to that of deuterons must arise mainly from the centrifugal force. However, the planned installation of an ITER-like first wall will result in future JET plasmas containing significant numbers of tungsten (W) ions, which have very high mass number (184) and are generally only partially ionized at typical JET temperatures and densities [12]. In these circumstances the Coriolis correction to the vertical field may, depending on the precise value of the W charge state, become comparable to externally-applied vertical fields.

In MAST $B \simeq 0.5\,\mathrm{T}$, $I_p \simeq 1\,\mathrm{MA}$, $T_i \lesssim 1\,\mathrm{keV}$ and $\Omega \lesssim 4 \times 10^5\,\mathrm{rads\ s^{-1}}$ [10]. At the highest rotation rates the relative change in the effective poloidal field arising from the Coriolis force is rather higher than it is in JET, and could be significant even for deuterons. Since, unlike the centrifugal force, the Coriolis force depends on the direction of rotation, one would expect to observe significant differences between the transport properties of MAST plasmas that are co-rotating and counter-rotating with respect to the plasma current. Indeed test-particle simulations of collisional W transport in MAST-like plasmas indicate that the particle confinement time in counter-rotating plasmas can exceed that in co-rotating plasmas by up to a factor of ten, depending on the W charge state [12]. Turbulent transport in rapidly-rotating MAST plasmas could be studied in the framework of either gyrokinetic or two-fluid theory, using appropriately defined equivalent fields in differentially-rotating co-moving frames.

With regard to this last point, it is important to note that the electric potential in the laboratory frame Φ_{lab} depends in general on the flow itself. For the purpose of evaluating Φ_* it is useful to have simple relations between Φ_{lab} and $\mathbf{u}_f$. Such relations can be obtained analytically from the equilibrium fluid equations for a dissipationless axisymmetric electron-ion plasma, neglecting momentum sources and assuming purely toroidal rotation [17]. Two limiting cases can be identified: rigid body rotation, such that Ω depends only on poloidal flux Ψ; and Keplerian rotation, in which the mechanical toroidal angular momentum per unit mass of the ion fluid is a flux function. In the rigid body case Φ_{lab} is related to Ω by the expression [18]

$$e\Phi_{\mathrm{lab}} = e\Phi_0(\Psi) + \frac{T_e}{2(T_e+T_i)} m_i \Omega(\Psi)^2 R^2, \tag{61}$$

where Φ_0 is a flux function and m_i is bulk ion mass. In the Keplerian case one obtains an expression of the form [17]

$$e\Phi_{\mathrm{lab}} = e\Phi_0(\Psi) - \frac{T_e}{2(T_e+T_i)} \frac{m_i \lambda(\Psi)^2}{R^2}, \tag{62}$$

where $\lambda(\Psi) = R^2\Omega$. Thus, in both cases Φ_{lab} is not a pure flux function. It is also possible to obtain relations between the flow and the temperature profiles, and between the flow and the plasma density. For both rigid body and Keplerian rotation the plasma density is higher on the outboard side of a flux surface than it is on the inboard side, due to centrifugal effects, although the dependence of the density on R differs in the two rotation scenarios [17]. Of course, the profiles of these quantities in real tokamaks depend in part on angular momentum sources and transport processes, but the idealised limiting cases considered in Ref. [17] serve to illustrate the processes whereby the potential can be determined self-consistently in terms of the flow and other plasma parameters. Having obtained $\Phi_{\rm lab}$ it is then straightforward to calculate Φ_f and proceed with the desired calculation (single-particle orbit, kinetic or fluid) in the non-inertial frame.

The frame flow $\mathbf{u}_f$ may not be divergence-free. However, in general we may decompose it into an incompressible component $\mathbf{u}_f^*$ and an irrotational component $\nabla\lambda_f$ where λ_f is a scalar function, i.e.

$$\mathbf{u}_f \;=\; \mathbf{u}_f^* + \nabla\lambda_f, \tag{63}$$

where

$$\nabla\cdot\mathbf{u}_f^* \;=\; 0, \tag{64}$$

$$\nabla\times\mathbf{u}_f^* \;=\; \mathbf{W}_f, \tag{65}$$

and

$$\nabla^2\lambda_f \;=\; \nabla\cdot\mathbf{u}_f. \tag{66}$$

If the flow is steady the particle Lagrangian does not depend on $\mathbf{V}\cdot\nabla\lambda_f$, since this term can be written as a total time derivative of λ_f and can thus be eliminated from the action. The Lagrangian takes the form

$$\begin{aligned}\mathcal{L}^* \;=\; & \frac{1}{2}mV^2 + Ze\mathbf{V}\cdot\left(\mathbf{A}+\frac{m}{Ze}\mathbf{u}_f^*\right)\\ & -Ze\left[\Phi_f - \frac{m}{2Ze}\left\{(u_f^*)^2+(\nabla\lambda_f)^2\right\} - \frac{m}{Ze}\nabla\cdot(\lambda_f\mathbf{u}_f^*)\right].\end{aligned} \tag{67}$$

The canonical momenta depend only upon the solenoidal component $\mathbf{u}_f^*$ and the effective vector potential $\mathbf{A}_* = \mathbf{A} + (m/Ze)\mathbf{u}_f^*$ is divergence-free if Φ_f is time-independent, in which case

$$\Phi_* \;=\; \Phi_f - \frac{m}{2Ze}\left\{(u_f^*)^2+(\nabla\lambda_f)^2\right\} - \frac{m}{Ze}\nabla\cdot(\lambda_f\mathbf{u}_f^*). \tag{68}$$

Thus the effective vector potential (and hence the effective magnetic field) in the non-inertial frame depend only on the solenoidal (incompressible) part of the frame flow which determines its vorticity, while the effective electric potential depends on both the solenoidal and irrotational parts of the flow. Hence a purely irrotational frame flow does not change the effective magnetic field, whereas a purely solenoidal flow modifies both the effective magnetic and effective electric fields.

We note a possible general application of potential interest in both tokamak and astrophysical applications. The fact that $\mathbf{u}_f(\mathbf{x},t)$ is non-relativistic but otherwise arbitrary can be exploited as follows: given $\mathbf{E}_{\rm lab}(\mathbf{x},t)$ and $\mathbf{B}_{\rm lab}(\mathbf{x},t)$ we could define a velocity field $\mathbf{u}_f$ by the three equations

$$m\left(\frac{\partial\mathbf{u}_f}{\partial t} + \mathbf{u}_f\cdot\nabla\mathbf{u}_f\right) \;=\; Ze\left(\mathbf{E}_{\rm lab}+\mathbf{u}_f\times\mathbf{B}_{\rm lab}\right). \tag{69}$$

The quantity $\mathbf{u}_f$ represents the fluid velocity of a species which is cold and dissipationless, and whose motion is governed by the electromagnetic fields in the laboratory frame. Writing the particle velocity in this frame as $\mathbf{v} = \mathbf{u}_f + \mathbf{V}$, we can obtain $\mathbf{V}$ by solving Eqs. (37-39). If, at each instant, we co-evolve $\mathbf{u}_f$ and $\mathbf{V}$ for a set of particles, we can obtain the particle trajectories in the laboratory frame by solving

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}_f(\mathbf{x}, t) + \mathbf{V}.$$

The velocity $\mathbf{u}_f$ is determined by $\mathbf{E} \times \mathbf{B}$ and polarization drifts; it does not contain the particle velocity since it is a frame velocity. Computing the particle motions after first subtracting off the electric field drifts does not necessarily lead to more accurate results than would be obtained by solving the equations of motion in the laboratory frame, but it may be advantageous to use the frame defined by Eq. (69) for the purpose of gyrophase averaging. Moreover, in the case of ions the equivalent fields $\mathbf{E}_*$ and $\mathbf{B}_*$ can depend significantly on $\mathbf{u}_f$ and hence, via Eq. (69), are influenced by spatial and temporal gradients in the laboratory frame fields. In these circumstances it may be easier to gain an intuitive understanding of the dynamics of charged particles by computing their orbits in a frame moving at velocity $\mathbf{u}_f$.

It should also be noted that the rotating frame equations are well-behaved for all values of the rotation parameter, ρ_Ω^*; taking the limit as this tends to zero yields the standard laboratory frame equations. The limit $\rho_\Omega^* \to \infty$ corresponds to either the ambient magnetic field or the charge-to-mass ratio going to zero. The drift ordering breaks down in this limit, but the exact Vlasov and Fokker-Planck equations remain entirely regular and describe the evolution in phase space of unmagnetized species. Gyrokinetic ordering in the equivalent fields requires $\omega_c^* \gg |\dot{E}_*/E_*|$, $|\dot{B}_*/B_*|$; and $r_L^*[|\nabla E_*/E_*| + |\nabla B_*/B_*|] \ll 1$. While these conditions may not always be satisfied, the exact equations can always be transformed to any non-relativistic co-moving frame using the analysis presented here.

VI. Discussion

We now relate our work to a paper on nonlinear gyrokinetics in rotating axisymmetric plasmas (an important special class of non-inertial frames) by Artun and Tang [19]. These authors consider frame toroidal flow velocities $\mathbf{u}_f = \omega_R(\Psi)R^2\nabla\phi$, and derive from a transformed Vlasov equation (Eq. (12) in Ref. [19]) a gyrokinetic equation. They partially follow our procedure by introducing a shift of origin in velocity space, $\mathbf{v}_{\mathrm{rot}} = \mathbf{v}_{\mathrm{lab}} - \mathbf{u}_f$ (note that our $\mathbf{u}_f = \mathbf{V}$ of Artun and Tang), but retain laboratory frame position coordinates $\mathbf{x}_{\mathrm{lab}}$. As we have shown, both our kinematic approach and the much more general Hamilton's principle deliver expressions of definitive simplicity for both the Vlasov equation and the Newton-Lorentz equations in $\mathbf{K}_f$. Indeed, the Vlasov equation employed by Artun and Tang contains, in addition to the transformed Maxwell fields, velocity-dependent terms and $\partial/\partial\mathbf{x}$, the gradient operator in the laboratory frame rather than the rotating frame.

The results derived by these authors should agree with any obtained from our equations, if carried to all orders, since both sets of equations are obtained from the inertial frame equations by purely mathematical manipulations. However, our equations exhibit a far simpler structure and reveal real physical characteristics of the motion which are obscured by the appearance of $\nabla \mathbf{u}_f$ which appears in all the formulae of Artun and Tang. We have shown that in the non-inertial frame the gyromotion is about the equivalent field $\mathbf{B}_*$ and adiabatic invariants are also defined with respect to this field. This simplicity is due to the well-known general covariance of Lagrange's equations under arbitrary transformations to generalized coordinates and the relativistic invariance of $A_\mu v^\mu$. In all formulations where our transformations are not used to transform to a co-moving non-inertial frame, one will find, in addition to the Lorentz forces in the frame, inertial forces depending upon the frame velocity and its spatial gradients in complicated ways, and also depending on the particle velocity with no simple physical interpretation. Of course, one is free to use any coordinate system (including the inertial laboratory frame) to solve problems, but we believe that our approach is more general and offers both rigor and intuitive appeal and should be easier to implement in full-orbit or gyro-orbit codes.

Next we discuss the relationship between the present analysis and previous work by Brizard [3]. Some general consequences which follow will also be stated. Brizard [3] considered the derivation of the nonlinear gyrokinetic equation in a toroidally co-rotating frame for the case of an axisymmetric tokamak plasma. Such frames, of course, constitute an important subset of the general class of non-inertial frames considered in the present paper. To clarify the nomenclature, the following conventions are noted: the laboratory frame coordinate and velocity variables in Brizard's paper are $(\mathbf{x}, \mathbf{v})$ and the plasma flow velocity is denoted by $\mathbf{u}$; this is equivalent to our $\mathbf{u}_f$. Brizard's non-inertial frame coordinate and velocity variables are $(\mathbf{r}, \mathbf{c})$; $\mathbf{c}$ is thus equivalent to our $\mathbf{V}$. Brizard proceeds to derive the gyrokinetic equation in the non-inertial frame using a phase-space Lagrangian Lie-transform perturbation method. Although he uses an action principle formalism in the non-relativistic limit, he does not discuss the transformation to the non-inertial frame of the Newton-Lorentz equations or the associated Vlasov equation, as in Sec. III of the present paper.

Within a self-consistent ordering expansion Brizard uses a first-order approximation to $\mathbf{u}$, namely $\mathbf{u}_0 = u_{0\|}\mathbf{b}+(1/B)\mathbf{b}\times\nabla\Phi_0$ where Φ_0 is a flux function, so that $\mathbf{b}\cdot\nabla\Phi_0 = 0$. He also introduces a particle velocity $\mathbf{u}_0^* \equiv \mathbf{u}_0 + W\mathbf{b}$, where W is the particle velocity parallel to $\mathbf{B}$, and an effective magnetic field

$$\mathbf{B}^* \equiv \nabla \times \mathbf{A}^* = \mathbf{B} + \frac{m}{Ze}\nabla \times \mathbf{u}_0^*. \tag{70}$$

This depends on both the particle's position and its velocity in the direction parallel to the magnetic field in the laboratory frame. It differs from our equivalent field $\mathbf{B}_*$, which, for a given flow $\mathbf{u}_f$, depends on particle position only. As we have shown, when viewed in a non-inertial frame a charged particle gyrates with respect to $\mathbf{B}_*$ rather than $\mathbf{B}$(or $\mathbf{B}^*$). The consequences of Brizard's analysis are most clearly exhibited in his Eqs. (18-19) for the drift orbit of the charged particle in the co-moving frame:

$$\dot{\mathbf{X}} = \frac{\mathbf{b}}{eB_\|^*} \times \nabla H + \frac{\mathbf{B}^*}{mB_\|^*}\frac{\partial H}{\partial W}, \tag{71}$$

$$\dot{W} = -\frac{\mathbf{B}^*}{mB_\|^*} \cdot \nabla H, \tag{72}$$

where $\mathbf{X}$ is the guiding center position,

$$H = e\Phi + \mu B + \frac{1}{2}m\mathbf{u}_0^* \cdot \mathbf{u}_0^*, \tag{73}$$

is the guiding-center Hamiltonian and $B_\parallel^*$ is the component of $\mathbf{B}^*$ parallel to $\mathbf{B}$, i.e.

$$B_\parallel^* = B\left(1 + \frac{\mathbf{b}}{\Omega} \cdot \nabla \times \mathbf{u}_0^*\right). \tag{74}$$

Comparing Eq. (71) with Eq. (49) we note two key differences. First, the "parallel" component of $\dot{\mathbf{X}} \equiv \mathbf{V}_{\rm gc}$ in Eq. (71) is taken along $\mathbf{B}^*$ rather than $\mathbf{B}$ or $\mathbf{B}_*$. Second, the perpendicular component is orthogonal to $\mathbf{B}$ rather than $\mathbf{B}_*$; the exact Lorentz-Newton equations in the non-inertial frame [Eq. (37)] indicate that $\mathbf{B}_*$ should be regarded as the true effective field in this frame. As we have noted, all particle drifts and adiabatic invariants should be defined with respect to the equivalent fields in $\mathbf{K}_f$. The physical reason for this is straightforward: in the laboratory frame the motion of a charged particle is determined solely by the $\mathbf{E}$ and $\mathbf{B}$ fields satisfying Maxwell's equations. In any inertial frame moving with constant velocity $\mathbf{u}$ relative to the laboratory, the motion is determined in the non-relativistic limit by the potentials $\mathbf{A}_{\rm lab}$ and $\Phi_{\rm lab} - \mathbf{A}\cdot\mathbf{u}$. When the co-moving frame is an accelerating one, inertial terms contribute to both $\mathbf{A}_*$ and Φ_*. In these circumstances it is natural and more accurate to use $\mathbf{B}_*$ as the magnetic field in an orbit theory calculation, even if the differences between $\mathbf{B}$, $\mathbf{B}^*$ and $\mathbf{B}_*$ are small.

As noted by Peeters and co-workers [4], it is appropriate to use a co-moving frame when applying gyrokinetic theory to rapidly rotating tokamak plasmas, since in these circumstances the electric field in the laboratory frame does not satisy the usual gyrokinetic ordering. However, we believe that the approach outlined in Sec. IV of the present paper provides a more physically-appealing route to the formulation of gyrokinetic theory in a non-inertial frame than that employed by Brizard. Subject to drift ordering being applicable in the non-inertial frame, our guiding center equations are applicable for arbitrary non-relativistic frame flows, including strongly sheared, non-neoclassical poloidal flows.

It should be noted that the motions of electrons in a frame co-rotating with a tokamak plasma are determined by Φ_f and $\mathbf{A}$, since in this case the modifications to the potentials arising from the frame flow are invariably negligible. The fact that the bulk ion motions in the rotating frame are determined by Φ_* and $\mathbf{A}_*$ rather than the Maxwell fields, combined with the need to maintain quasi-neutrality, leads to the self-consistent expressions for $\Phi_{\rm lab}$ given by Eqs. (61) and (62) [17]. Charged particle trajectories in the laboratory frame must, of course, be independent of the frame in which the equation of motion is solved; this frame-independence is guaranteed by the use of the exact Lagrangian given by Eq. (28). Since, in a non-inertial frame, $\mathbf{A}_*$ depends on the particle charge-to-mass ratio, the effective flux surfaces also depend on Ze/m [12]. One consequence of this is that gyrokinetic simulation codes which model transport within flux tubes rather than the entire plasma should employ different flux tube geometries for electrons and bulk ions if the gyrokinetic equations for these species are solved in a co-moving frame. However, there is no difficulty in using our formalism to obtain the required equations.

VII. Conclusions

Using Hamilton's principle of least action we have demonstrated that the equation of motion of a charged particle in a non-relativistic but otherwise arbitrary non-inertial frame has the same form as it does in an inertial frame, with effective electric and magnetic fields that depend on both the Maxwell fields in the non-inertial frame and the frame flow, $\mathbf{u}_f$. The sole effect of the Coriolis force is to introduce an additional term in the effective vector potential, $(m/Ze)\mathbf{u}_f$. In the case of a tokamak plasma rotating toroidally at a constant angular velocity Ω which is either uniform or depends only on R, this gives rise to an additional vertical field. The Maxwell electric potential in the non-inertial frame differs from the inertial frame potential by $-\mathbf{A}\cdot\mathbf{u}_f$ where $\mathbf{A}$ is the Maxwell vector potential (which is frame-independent); the effective potential Φ_* differs from the Maxwell potential in the non-inertial frame by $mu_f^2/(2Ze)$. The effective vector potential depends only on the solenoidal (incompressible) part of the frame flow, whereas the effective electric potential depends on both the solenoidal and irrotational parts of $\mathbf{u}_f$. The Vlasov, Fokker-Planck, Langevin, drift-orbit, drift-kinetic, gyrokinetic and fluid equations can be formally derived exactly as in the case of an inertial frame, except that the effective fields and potentials must be used. A dissipationless two-fluid model developed by the present authors can be used to relate the toroidal flow in a tokamak plasma to the equilibrium laboratory frame electric potential; the corresponding effective fields in the co-moving frame can then be easily calculated. Our analysis is computationally testable under tokamak plasma conditions, and can be readily extended to include relativistic flows and spacetime curvature, thereby making it applicable to extreme astrophysical plasma environments, such as the magnetospheres of rapidly-rotating pulsars. Although we have given particular attention to rotating frames, the analysis may also have applications to plasmas undergoing radial acceleration, for example those in inertial confinement fusion experiments [20] or supernova explosions [21].

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